

FJRW-RINGS AND MIRROR SYMMETRY

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ABSTRACT. The Landau-Ginzburg Mirror Symmetry Conjecture states that for a quasi-homogeneous singularity W and a group G of symmetries of W , there is a dual singularity W^T such that the orbifold A-model of W/G is isomorphic to the B-model of W^T . The Landau-Ginzburg A-model is the Frobenius algebra $\mathcal{H}_{W,G}$ constructed by Fan, Jarvis, and Ruan, and the B-model is the orbifold Milnor ring of W^T . We verify the Landau-Ginzburg Mirror Symmetry Conjecture for Arnol'd's list of unimodal and bimodal quasi-homogeneous singularities with G the maximal diagonal symmetry group, and include a discussion of eight axioms which facilitate the computation of FJRW-rings.

1. INTRODUCTION

In this paper we verify the Landau-Ginzburg Mirror Symmetry Conjecture for Arnol'd's list of unimodal and bimodal singularities [1, pg 25]. Briefly, the conjecture states that for non-degenerate quasi-homogeneous singularities there is a mirror dual singularity such that the ring constructed by Fan-Jarvis-Ruan [4] for one is isomorphic to the Landau-Ginzburg B-model (Milnor ring) of the other. The conjecture has already been proven for the simple and parabolic singularities in [4].

The Landau-Ginzburg B-model is an orbifolded Milnor ring. When the orbifold group is trivial, this is just the classical Milnor ring (local algebra) of the singularity. [1]

In [4], Fan, Jarvis, and Ruan construct a cohomological field theory which gives the A-model Frobenius algebra when restricted to genus zero with three marked points. Since the original motivation for the theory was to study generalizations of the Witten equation, we call the A-model Frobenius Algebra the Fan-Jarvis-Ruan-Witten ring, or FJRW-ring, for short.

The singularities for this particular theory are required to be non-degenerate, quasi-homogeneous (i.e. weighted homogeneous) polynomials, having an isolated singularity at the origin. Not all of the singularities in the list in [1] are quasi-homogeneous, so we have only used those which are. Several of the families in [1] depend on certain parameters, but are only non-degenerate and quasi-homogeneous for particular parameter values. In these cases, we fix the appropriate parameter values without further comment.

1.1. Outline of Paper.

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1.2. Review of Construction. Let W be a non-degenerate quasi-homogeneous polynomial in the variables x_1, x_2, \dots, x_N with weights q_1, q_2, \dots, q_N respectively. Non-degeneracy requires that these weights are uniquely determined by the condition that each monomial in W has total weight 1, and that W has an isolated singularity at the origin.

To each quasi-homogeneous polynomial W we can associate a matrix, B_W , such that the columns correspond to the variables, and the rows correspond to the terms of the polynomial (see [2]). In other words, the entry $(B_W)_{i,j}$ is the power of x_j in the i -th monomial of W . If the number of monomials coincides with the number of variables, the matrix B_W is square, and the non-degeneracy condition implies that B_W is invertible. In such cases, we call the polynomial W an *invertible potential*. Note that the variables of an invertible potential can always be rescaled so the coefficients of the monomials are all equal to one.

To illustrate the correspondence $W \leftrightarrow B_W$, consider the polynomial $W = x^3 + xy^4 + yz^2$. The corresponding matrix is

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

If the matrix B_W is square, its transpose B_W^T will also correspond to a quasi-homogeneous polynomial, which we denote by W^T . In many cases W^T polynomial also has an isolated singularity at the origin, thus satisfying the non-degeneracy condition.

The central charge of W is defined to be

$$\hat{c} := \sum_{j=1}^N (1 - 2q_j).$$

The Jacobian ideal \mathcal{J} is defined by

$$\mathcal{J} = \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_N} \right).$$

The Milnor ring \mathcal{Q}_W is given by

$$\mathcal{Q}_W := \mathbb{C}[x_1, x_2, \dots, x_N] / \mathcal{J}$$

together with the residue pairing. \mathcal{Q}_W is a finite dimensional vector space over \mathbb{C} , with dimension

$$\mu = \prod_{j=1}^N \left(\frac{1}{q_j} - 1 \right).$$

It is graded by weighted degree, and the elements of top degree form a one-dimensional subspace generated by $\text{hess}(W) = \det \left(\frac{\partial^2 W}{\partial x_i \partial x_j} \right)$. One can check directly that the top degree is equal to \hat{c} .

For $f, g \in \mathcal{Q}_W$, the residue pairing $\langle f, g \rangle$ may be defined by

$$fg = \frac{\langle f, g \rangle}{\mu} \text{hess}(W) + \text{lower order terms}.$$

This pairing is non-degenerate, and endows the Milnor ring with the structure of a Frobenius algebra.

To define the FJRW ring, we require in addition to W a choice of a group of diagonal symmetries of W . The choice of group heavily affects the resulting structure of the FJRW ring. The maximal group of diagonal symmetries is defined as

$$G_W = \{ (\alpha_1, \alpha_2, \dots, \alpha_N) \subseteq (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_N x_N) = W(x_1, x_2, \dots, x_N) \}$$

Note that G_W always contains the exponential grading element $J = (e^{2\pi i q_1}, e^{2\pi i q_2}, \dots, e^{2\pi i q_N})$. In general, the theory requires that the symmetry group be *admissible* (see [4] section 2.3). In our computations, we will use either the maximal diagonal symmetry group G_W or the subgroup generated by J ; both of these groups known to be admissible.

The Landau-Ginzburg Mirror Symmetry Conjecture states the following:

Conjecture. *For a non-degenerate, quasi-homogeneous, invertible singularity W and (maximal) diagonal symmetry group G , there is a dual singularity W^T so that the FJRW-ring of W/G is isomorphic to the (unorbifolded) Milnor ring of W^T .*

Remark. We use the notation W^T suggestively for the dual singularity, as we identify in this paper a class of examples for which the Berglund-Hübsch transposed singularity is the appropriate dual in the context of the Landau-Ginzburg Mirror Symmetry Conjecture.

We now outline the definition of $\mathcal{H}_{W,G}$ as a \mathbb{C} -vector space, after which we will define the pairing, grading, and multiplication that make $\mathcal{H}_{W,G}$ a Frobenius algebra.

In [4], the state space $\mathcal{H}_{W,G}$ is defined in terms of Lefschetz thimbles. For computational convenience, we give a presentation in terms of Milnor rings, but we should point out that the isomorphism between the two presentations is not canonical.

Let G be an admissible group. For $h \in G$, let $\text{Fix } h \subset \mathbb{C}^N$ be the fixed locus of h , and let N_h be its dimension. Define

$$\mathcal{H}_h := \Omega^{N_h}(\mathbb{C}^{N_h}) / (dW|_{\text{Fix } h} \wedge \Omega^{N_h-1}) \cong \mathcal{D}_{W|_{\text{Fix } h}} \cdot \omega$$

where $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{N_h}}$ is the natural choice of volume form.

G acts on \mathcal{H}_h via its action on the coordinates, and the state space of the FJRW-ring is the vector space of invariants under this action, i.e.

$$\mathcal{H}_{W,G} := \left(\bigoplus_{h \in G} \mathcal{H}_h \right)^G.$$

$\mathcal{H}_{W,G}$ is \mathbb{Q} -graded by the so-called W -degree, which depends only on the G -grading. To define this grading, first note that each element $h \in G$ can be uniquely expressed as

$$h = (e^{2\pi i \Theta_1^h}, e^{2\pi i \Theta_2^h}, \dots, e^{2\pi i \Theta_N^h})$$

with $0 \leq \Theta_i^h < 1$.

For $\alpha_h \in (\mathcal{H}_h)^G$, the W -degree of α_h is defined by

$$\deg_W(\alpha_h) := N_h + 2 \sum_{j=1}^N (\Theta_j^h - q_j). \quad (1)$$

Since $\text{Fix } h = \text{Fix } h^{-1}$, we have $\mathcal{H}_h \cong \mathcal{H}_{h^{-1}}$, and the pairing on $\mathcal{D}_{W|_{\text{Fix } h}}$ induces a pairing

$$(\mathcal{H}_h)^G \otimes (\mathcal{H}_{h^{-1}})^G \rightarrow \mathbb{C}.$$

The pairing on $\mathcal{H}_{W,G}$ is the direct sum of these pairings. Fixing a basis for $\mathcal{H}_{W,G}$, we denote the pairing by a matrix $\eta_{\alpha,\beta} = \langle \alpha, \beta \rangle$, with inverse $\eta^{\alpha,\beta}$.

For each pair of non-negative integers g and n , with $2g - 2 + n > 0$, the FJRW cohomological field theory produces classes $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_n) \in H^*(\overline{\mathcal{M}}_{g,n})$ of complex codimension D for each n -tuple $(\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathcal{H}_{W,G})^n$. The codimension D is given by

$$D := \hat{c}_W(g-1) + \frac{1}{2} \sum_{i=1}^n \deg_W(\alpha_i),$$

and the n -point correlators are defined to be

$$\langle \alpha_1, \dots, \alpha_n \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}^W(\alpha_1, \dots, \alpha_n).$$

The correlator $\langle \alpha_1, \dots, \alpha_n \rangle_{g,n}$ vanishes unless the codimension of $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_n)$ is zero. The ring structure on $\mathcal{H}_{W,G}$ is determined by the genus-zero three-point correlators. In other words, if $r, s \in \mathcal{H}_{W,G}$, then

$$r * s := \sum_{\alpha, \beta} \langle r, s, \alpha \rangle_{0,3} \eta^{\alpha, \beta} \beta \quad (2)$$

where the sum is taken over all choices of α and β in a fixed basis of $\mathcal{H}_{W,G}$.

The classes $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_n)$ satisfy the following axioms that allow us to compute most of the three-point correlators $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ explicitly.

Axiom 1. Dimension: *If $D \notin \frac{1}{2}\mathbb{Z}$, then $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$. Otherwise, D is the complex codimension of the class $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_n)$. In particular, if $g = 0$ and $n = 3$, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 0$ unless $D = 0$.*

Notice that in the case where $g = 0$ and $n = 3$, $D = 0$ if and only if $\sum_{i=1}^3 \deg_W \alpha_i = 2\hat{c}$.

Axiom 2. Symmetry: *Let $\sigma \in S_n$. Then*

$$\langle \alpha_1, \dots, \alpha_n \rangle_{g,n} = \langle \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)} \rangle_{g,n}.$$

The next few axioms rely on the degrees of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_N$ endowing an orbicurve with a so-called *W-structure*; however, this can be reduced to a simple numerical criterion. Consider the class $\Lambda_{g,k}^W(\alpha_1, \alpha_2, \dots, \alpha_k)$, with $\alpha_j \in (\mathcal{H}_{h_j})^G$ for each $j \in \{1, \dots, N\}$. For each variable x_j , define l_j by

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^k \Theta_j^{h_i}$$

Axiom 3. Integer degrees: *If $l_j \notin \mathbb{Z}$ for some $j \in \{1, \dots, N\}$, then $\Lambda_{g,k}^W(\alpha_1, \alpha_2, \dots, \alpha_k) = 0$.*

Axiom 4. Concavity: *If $l_j < 0$ for all $j \in \{1, 2, 3\}$, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 1$.*

The next axiom is related to the Witten map:

$$\mathcal{W} : \bigoplus_{j=1}^N \mathbb{C}^{h_j^0} \rightarrow \bigoplus_{j=1}^N \mathbb{C}^{h_j^1}$$

$$\mathcal{W} = \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_N} \right)$$

where h_j^0 and h_j^1 are defined by

$$h_j^0 := \begin{cases} 0 & \text{if } l_j < 0 \\ l_j + 1 & \text{if } l_j \geq 0 \end{cases}$$

$$h_j^1 := \begin{cases} -l_j - 1 & \text{if } l_j < 0 \\ 0 & \text{if } l_j \geq 0 \end{cases}$$

so that both are non-negative integers satisfying $h_j^0 - h_j^1 = l_j + 1$.*

The fact that the Witten map is well-defined is a consequence of the geometric conditions on the \mathcal{L}_j considered in [4]. For further details, we refer readers to the original paper.

If $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_n)$ is a class of codimension zero, we obtain a complex number by integrating over $\overline{\mathcal{M}}_{g,n}$. Abusing notation, we will refer to the class $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_n)$ and its integral over $\overline{\mathcal{M}}_{g,n}$ interchangeably.

*The reader may note that h_j^i is just the dimension of the i -th cohomology of the j -th line bundle in the W -structure, and this relation is just Riemann-Roch for a line-bundle L_j of degree l_j on \mathbb{CP}^1 . The preceding definitions simply serve to axiomatize the entire construction.

Axiom 5. *Index Zero:* Consider the class $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_n)$, with $\alpha_i \in \mathcal{H}_{\gamma_i, G}$. If $\text{Fix } \gamma_i = \{0\}$ for each $i \in \{1, 2, \dots, n\}$ and

$$\sum_{j=1}^N (h_j^0 - h_j^1) = 0$$

then $\Lambda_{g,n}(\alpha_1, \alpha_2, \dots, \alpha_n)$ is of codimension zero and $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_n)$ is equal to the degree of the Witten map.

Axiom 6. *Composition:* If the four-point class, $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is of codimension zero, then it decomposes in terms of three-point correlators in the following way:

$$\Lambda_{0,4}^W(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{\beta, \delta} \langle \alpha_1, \alpha_2, \beta \rangle \eta^{\beta, \delta} \langle \delta, \alpha_3, \alpha_4 \rangle.$$

Note that $\text{Fix } J = \{0\}$ so $\mathcal{H}_J \cong \mathbb{C}$. The identity element in the FJRW-ring is an element of \mathcal{H}_J , and we denote this element by $\mathbb{1}$.

Axiom 7. *Pairing:* For $\alpha_1, \alpha_2 \in \mathcal{H}_{W,G}$, $\langle \alpha_1, \alpha_2, \mathbb{1} \rangle = \eta(\alpha_1, \alpha_2)$, where η is the pairing in $\mathcal{H}_{W,G}$.

Axiom 8. *Sums of singularities:* If $W_1 \in \mathbb{C}[x_1, \dots, x_r]$ and $W_2 \in \mathbb{C}[y_1, \dots, y_s]$ are two non-degenerate, quasi-homogeneous polynomials with maximal symmetry groups G_1 and G_2 , then the maximal symmetry group of $W = W_1 + W_2$ is $G = G_1 \times G_2$, and there is an isomorphism of Frobenius algebras

$$\mathcal{H}_{W,G} \cong \mathcal{H}_{W_1, G_{W_1}} \otimes \mathcal{H}_{W_2, G_{W_2}}$$

Remark. We note an important consequence of Axiom 8. Under the same hypotheses as in the statement of the axiom, we have a Frobenius Algebra isomorphism

$$\mathcal{Q}_W \cong \mathcal{Q}_{W_1} \otimes \mathcal{Q}_{W_2},$$

and similarly

$$\mathcal{Q}_{W^T} \cong \mathcal{Q}_{W_1^T} \otimes \mathcal{Q}_{W_2^T}.$$

Consequently, in order to prove the Mirror Symmetry Conjecture for $W = W_1 + W_2$, a sum of decoupled polynomials (with maximal A -model orbifold group), it suffices to prove it for W_1 and W_2 individually.

These axioms allow us to compute most of the three-point correlators of the FJRW-rings. In some cases, the axioms are not enough to compute all of the correlators; however, in most of these cases, one can still verify the mirror symmetry conjecture.

1.3. Additional Notation. A singularity is said to be *invertible* if the number of monomials equals the number of variables. We have found that the Landau-Ginzburg mirror symmetry conjecture holds for the invertible unimodal and bimodal singularities orbifolded by the maximal group of diagonal symmetries, G_W . The conjecture was verified in [4] for the simple singularities.

In most cases considered, the maximal symmetry group, G_W , is cyclic. In these cases we have adopted the following notation. Let g be a generator for G_W . If J generates G_W , take $g = J$. If $\text{Fix } g^k = \{0\}$, define

$$e_k = 1 \in \mathcal{H}_{g^k} \cong \mathbb{C},$$

otherwise if $\text{Fix } g^k = \mathbb{C}x_{i_1} \oplus \dots \oplus \mathbb{C}x_{i_{N_g}}$ define

$$e_k = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_{N_g}} \in \mathcal{H}_{g^k}.$$

We denote coordinate subspaces of \mathbb{C}^N with subscripts indicating the non-zero variables in these subspaces. So, for example, the xy -plane in \mathbb{C}^3 will be denoted by \mathbb{C}_{xy}^2 .

Our computations are often made easier by judicious use of associativity. This will be reflected typographically in the grouping of terms. So, for example,

$$\alpha * \alpha\beta$$

indicates that $\alpha\beta$ should be computed first, and then multiplied by α .

1.4. An example. We will now give an example demonstrating the construction and our methods of computation more fully.

1.4.1. E_{19} with maximal symmetry group. The polynomial for E_{19} is $x^3 + xy^7$. The corresponding weights for each variable are $q_x = \frac{1}{3}, q_y = \frac{2}{21}$ and the central charge is $\hat{c} = \frac{24}{21}$. The Jacobian ideal is $\mathcal{J} = (3x^2 + y^7, 7xy^6)$.

The maximal group of diagonal symmetries is given by

$$G = \langle (\alpha, \beta) \mid \alpha^3, \alpha\beta^7 \rangle$$

From the relations, we can see that $\alpha = \beta^{-7}$, so that $|G| = 21$ and G is cyclic. The exponential grading element is $J = (e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{21}})$, which has order 21, so the maximal group of diagonal symmetries is generated by J .

The fixed point locus of J^k is given by

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } 3 \nmid k, k \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The Milnor ring is $\mathbb{C}[x, y]/\mathcal{J} \cong \langle 1, x, x^2, y, \dots, y^6, xy, xy^2, \dots, xy^5, x^2y, x^2y^2, \dots, x^2y^5 \rangle$ for $J^0 = I$. If $3 \mid k$, then $W|_{\text{Fix } J^k} = x^3$. And in the case that $3 \nmid k$, the Milnor ring is trivial. So the Milnor rings are given by

$$\mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, x, x^2, y, \dots, y^6, xy, xy^2, \dots, xy^5, x^2y, x^2y^2, \dots, x^2y^5 \rangle, \mu = 19 & \text{if } k = 0 \\ \langle 1, x \rangle, \mu = 2 & \text{if } 3 \nmid k, k \neq 0 \\ \langle 1 \rangle & \text{otherwise} \end{cases}$$

Now for each choice of k with $3 \nmid k$, $\text{Fix } J^k = \{0\}$ so $\mathcal{H}_{J^k, G} \cong \mathbb{C}$ with trivial G -action. So e_k is a generator for the state space. To take G -invariants, we need only consider the action of J , since $G = \langle J \rangle$.

If $k \neq 0$ and $3 \mid k$, then we must consider the action of J on $x^j dx$ for $j \in \{0, 1\}$.

$$J \cdot x^j dx = \exp\left(2\pi i \frac{(j+1)}{3}\right) x^j dx$$

This action clearly has no invariants.

For $k = 0$, we must consider the action of J on the Milnor ring associated to J^0 .

$$J \cdot x^l y^m dx \wedge dy = \exp\left(2\pi i \frac{(7l+7+2m+2)}{21}\right) x^l y^m dx \wedge dy.$$

This action has a single invariant element, namely $y^6 dx \wedge dy = y^6 e_0$. So a basis for the state space is

$$\mathcal{H}_{E_{19}, \langle J \rangle} = \langle y^6 e_0, e_1, e_2, e_4, e_5, e_7, e_8, e_{10}, e_{11}, e_{13}, e_{14}, e_{16}, e_{17}, e_{19}, e_{20} \rangle.$$

The W -degrees are straightforward to compute from formula (1) on pg 3. Again, notice that the W -degree only depends on the power of the generator J . We give the invariants of each sector in the following table as well.

k	0	1	2	4	5	7	8	10	11	13	14	16	17	19	20
$ G \cdot \deg_W$	24	0	18	12	30	24	42	36	12	6	24	18	36	30	48
invariants	$y^6 e_0$	$\mathbb{1}$	e_2	e_4	e_5	e_7	e_8	e_{10}	e_{11}	e_{13}	e_{14}	e_{16}	e_{17}	e_{19}	e_{20}

Several three-point correlators can be computed using the pairing axiom. Note that all twisted sectors correspond to trivial fixed loci, so the residue pairing between twisted sectors is given by

$$\begin{aligned} \mathcal{H}_{J^k} \otimes \mathcal{H}_{J^{21-k}} &\xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C} \\ \langle e_k, e_{21-k} \rangle &= 1 \end{aligned}$$

So, by the pairing axiom, $\langle \mathbf{1}, \mathbf{1}, e_{20} \rangle$, $\langle \mathbf{1}, e_2, e_{19} \rangle$, $\langle \mathbf{1}, e_4, e_{17} \rangle$, $\langle \mathbf{1}, e_5, e_{16} \rangle$, $\langle \mathbf{1}, e_7, e_{14} \rangle$, $\langle \mathbf{1}, e_8, e_{13} \rangle$, and $\langle \mathbf{1}, e_{10}, e_{11} \rangle$ are all equal to 1.

The pairing in the untwisted sector gives us

$$\langle y^6 e_0, y^6 e_0, \mathbf{1} \rangle = \eta_{y^6 e_0, y^6 e_0} = \frac{19y^{12}}{\text{Hess}} = \frac{19y^{12}}{-133y^{12}} = -1/7.$$

The following three-point correlators have l_x and l_y both less than zero: $\langle e_2, e_4, e_{16} \rangle$, $\langle e_2, e_7, e_{13} \rangle$, $\langle e_4, e_4, e_{14} \rangle$, $\langle e_4, e_5, e_{13} \rangle$, $\langle e_4, e_7, e_{11} \rangle$, $\langle e_{11}, e_{13}, e_{19} \rangle$, $\langle e_{11}, e_{16}, e_{16} \rangle$, $\langle e_{13}, e_{13}, e_{17} \rangle$, $\langle e_{13}, e_{14}, e_{16} \rangle$. Therefore, by the concavity axiom each is equal to 1.

For the three-point correlator $\langle y^6 e_0, e_{11}, e_{11} \rangle$, we apply the composition axiom to the class $\Lambda_{0,4}^W(e_{11}, e_{11}, e_{11}, e_{11})$. The values for the line bundle degrees are $l_x = -2$ and $l_y = 0$. So we have $H^0 = 0 \oplus \mathbb{C}_y$ and $H^1 = \mathbb{C}_x \oplus 0$. The Witten map $H^0 \rightarrow H^1$ is given by the complex conjugate of the gradient of W . In other words it maps

$$(0, y) \mapsto (\overline{y}^7, 0).$$

The degree of this map is -7 , so the composition axiom tells us that

$$-7 = \Lambda_{0,4}^W(e_{11}, e_{11}, e_{11}, e_{11}) = \sum_{\alpha, \beta} \langle e_{11}, e_{11}, \alpha \rangle \eta^{\alpha, \beta} \langle \beta, e_{11}, e_{11} \rangle,$$

where α and β range over a fixed basis for the FJR-ring.

By degree considerations, the only non-zero contribution to this sum occurs when $\alpha = y^6 e_0 = \beta$. Setting $a := \langle y^6 e_0, e_{11}, e_{11} \rangle$, we get the following equation:

$$\begin{aligned} -7 &= \langle e_{11}, e_{11}, y^6 e_0 \rangle \eta^{y^6 e_0, y^6 e_0} \langle y^6 e_0, e_{11}, e_{11} \rangle \\ &= -7a^2, \end{aligned} \tag{3}$$

where the -7 on the right is the contribution to the inverse of the pairing corresponding to $\langle y^6 e_0, y^6 e_0, \mathbf{1} \rangle = -1/7$ computed above.

From this we can see $\langle y^6 e_0, e_{11}, e_{11} \rangle = \pm 1$. We will see below that either choice gives a Frobenius algebra that is isomorphic to $\mathcal{Q}_{E_{19}^T}$, where $E_{19}^T = x^3 y + y^7$.

This completes the computation of the three-point correlators. All others are required to vanish by Axioms 1 and 2.

Recall that multiplication in $\mathcal{H}_{E_{19}, G}$ is given by equation (2). Examining degrees we see that e_{13} is a generator for $\mathcal{H}_{E_{19}, G}$. We compute

$$\begin{aligned} e_{13}^2 &= \sum \alpha, \beta \langle e_{13}, e_{13}, \alpha \rangle \eta^{\alpha, \beta} \beta \\ &= \langle e_{13}, e_{13}, e_{17} \rangle \eta^{e_{17}, e_4} e_4 \\ &= e_4 \end{aligned}$$

where the first summation is taken over all α and β among the generators we have listed for the state space. The second equality follows because the only non-zero three-point correlator with e_{13} occurring twice is $\langle e_{13}, e_{13}, e_{17} \rangle = 1$. Again examining degrees, we see that e_{11} is also a generator of $\mathcal{H}_{E_{19}, G}$.

One can check directly that e_{13} and e_{11} generate $\mathcal{H}_{E_{19}, G}$ as a Frobenius algebra, so we may define a surjective map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{E_{19}, G}$ by $X \mapsto e_{11}$ and $Y \mapsto e_{13}$, and extend to a \mathbb{C} -algebra homomorphism.

We see that $e_{11}^2 * e_{13} = 0$, $e_{11}^3 = -7e_{10}$, and $e_{13}^6 = e_{10}$. Hence $(3X^2Y, X^3 + 7Y^6) \subset \ker \varphi$.

Therefore since $\mathbb{C}[X, Y]/(3X^2Y, X^3 + 7Y^6) = \mathcal{Q}_{E_{19}^T}$ has dimension 15, the same as $\mathcal{H}_{E_{19}, G}$, we deduce that the inclusion $(3X^2Y, X^3 + 7Y^6) \subset \ker \varphi$ is an equality, therefore the map induces a degree-preserving isomorphism $\mathcal{Q}_{E_{19}^T} \cong \mathcal{H}_{E_{19}, G}$.

1.5. Format of results. For each singularity, we will display the information in the following pattern:

- Name of singularity, the defining polynomial, the Jacobian ideal, the weights associated to each variable, and the central charge. We will also give the symmetry group used in the construction, typically G_W , which we will henceforth denote by G .
- Fixed locus for each group element.
- Basis for the Milnor ring of W restricted to each fixed locus.
- Table of sectors with non-trivial G -invariants, including the invariant elements and their W -degrees. For clarity of exposition, we will multiply W -degrees by a factor of $|G|$.
- We will give the values of the three-point correlators that are not required to vanish by Axioms 1 and 2. These will be grouped in the following order: those computed by the Pairing axiom, those by the Concavity axiom, those by the Index Zero axiom, and those by the Composition axiom. Any correlators for which the axioms do not suffice will be listed last, including any relations among them.
- Finally, we will describe, where possible, an isomorphism between the FJRW-ring of W and the Milnor ring of W^T .

2. COMPUTATIONS

We take our examples from the unimodal and bimodal singularities listed by Arnol'd [1]. Many of these singularities are quasi-homogeneous only after fixing specific parameter values, which we do without further comment.

2.1. Unimodal Singularities.

$\mathbf{Q}_{10} = \mathbf{x}^2\mathbf{z} + \mathbf{y}^3 + \mathbf{z}^4$. Axiom 8 applies here. By the subsequent comment, it suffices to prove the Mirror Symmetry Conjecture for $D_5 = x^2z + z^4$ and $A_2 = y^3$. The conjecture was proved for the simple (ADE) singularities in [4], so it holds in this case also.

$\mathbf{Q}_{11} = \mathbf{x}^2\mathbf{z} + \mathbf{y}^3 + \mathbf{yz}^3$.

$$\begin{aligned} \mathcal{J} &= (2xz, 3y^2 + z^3, x^2 + 3yz^2) & q_x &= \frac{7}{18}, q_y = \frac{6}{18}, q_z = \frac{4}{18}, \hat{c} = \frac{20}{18} & G &= \langle J \rangle = \mathbb{Z}/18\mathbb{Z} \\ \text{Fix } J^k &= \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } k = 3, 6, 12, 15 \\ \mathbb{C}_{yz}^2 & \text{if } k = 9 \\ 0 & \text{otherwise} \end{cases} & \mathcal{Q}_{|\text{Fix } J^k} &= \begin{cases} \langle 1, z, y, x, z^2, yz, z^3, xy, yz^2, z^4, z^5 \rangle, \mu = 11 \\ \langle 1, y, y^2 \rangle, \mu = 3 \\ \langle 1, z, y, z^2, yz, z^3, z^4 \rangle, \mu = 7 \\ \langle 1 \rangle \end{cases} \end{aligned}$$

k	1	2	4	5	7	8	9	10	11	13	14	16	17
$ G \cdot \deg_W$	0	34	30	28	24	22	20	18	16	12	10	6	40
invariants	$\mathbb{1}$	e_2	e_4	e_5	e_7	e_8	$z^2 e_9$	e_{10}	e_{11}	e_{13}	e_{14}	e_{16}	e_{17}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbb{1}, \mathbb{1}, e_{17} \rangle$, $\langle \mathbb{1}, e_2, e_{16} \rangle$, $\langle \mathbb{1}, e_4, e_{14} \rangle$, $\langle \mathbb{1}, e_5, e_{13} \rangle$, $\langle \mathbb{1}, e_7, e_{11} \rangle$, and $\langle \mathbb{1}, e_8, e_{10} \rangle$ are all equal to 1, and $\langle \mathbb{1}, z^2 e_9, z^2 e_9 \rangle = -\frac{1}{3}$.

By the concavity axiom, $\langle e_5, e_{16}, e_{16} \rangle$, $\langle e_7, e_{14}, e_{16} \rangle$, $\langle e_{10}, e_{11}, e_{16} \rangle$, $\langle e_{10}, e_{13}, e_{14} \rangle$, are all equal to 1.

By the index-zero axiom, $\langle e_{11}, e_{13}, e_{13} \rangle$ and $\langle e_8, e_{13}, e_{16} \rangle$ are -2 .

By the composition axiom,

$$-3 = \langle z^2 e_9, e_{14}, e_{14} \rangle \eta^{z^2 e_9, z^2 e_9} \langle z^2 e_9, e_{14}, e_{14} \rangle, \quad (\dagger)$$

where $\eta^{z^2 e_9, z^2 e_9} = -3$, so $\langle z^2 e_9, e_{14}, e_{14} \rangle = \pm 1$.

Consider the map

$$\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{Q_{11}, G}$$

defined by $X \mapsto e_{10}$, $Y \mapsto e_{14}$, and $Z \mapsto e_{16}$ extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_{16}^3 = -2e_{10}$, $e_{14}^3 = -3e_4 = -3e_{10} * e_{16}^2$, and $e_{14}^2 * e_{16} = 0$.

Hence $(2X + Z^3, 3Y^2 Z, Y^3 + 3XZ^2) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(2X + Z^3, 3Y^2 Z, Y^3 + 3XZ^2) = \mathcal{Q}_{Q_{11}^T}$ has dimension 13, we deduce the inclusion is in fact equality, and $\mathcal{H}_{Q_{11}, G} \cong \mathcal{Q}_{Q_{11}^T}$.

Q₁₂ = x²z + y³ + z⁵. Axiom 8 is applicable, and it suffices to prove the Mirror Symmetry Conjecture for $D_6 = x^2 z + z^5$ and $A_2 = y^3$. This follows from the study of simple singularities in [4].

S₁₁ = x²z + yz² + y⁴.

$$\mathcal{J} = (2xz, z^2 + 4y^3, x^2 + 2yz) \quad q_x = \frac{5}{16}, q_y = \frac{4}{16}, q_z = \frac{6}{16}, \hat{c} = \frac{18}{16} \quad G = \langle J \rangle = \mathbb{Z}/16\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}_{xy^2}^3 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } k = 4, 12 \\ \mathbb{C}_{yz}^2 & \text{if } k = 8 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, y, x, z, y^2, xy, x^2, z^2, xy^2, y^2 z, z^3 \rangle, \mu = 11 \\ \langle 1, z, z^2 \rangle, \mu = 3 \\ \langle 1, z, y, y^2, y^3 \rangle, \mu = 5 \\ \langle 1 \rangle \end{cases}$$

k	1	2	3	5	6	7	8	9	10	11	13	14	15
$ G \cdot \deg_W$	0	30	28	24	22	20	18	16	14	12	8	6	36
invariants	$\mathbb{1}$	e_2	e_3	e_5	e_6	e_7	ze_8	e_9	e_{10}	e_{11}	e_{13}	e_{14}	e_{15}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbb{1}, \mathbb{1}, e_{15} \rangle$, $\langle \mathbb{1}, e_2, e_{14} \rangle$, $\langle \mathbb{1}, e_3, e_{13} \rangle$, $\langle \mathbb{1}, e_5, e_{11} \rangle$, $\langle \mathbb{1}, e_6, e_{10} \rangle$, and $\langle \mathbb{1}, e_7, e_9 \rangle$ are all equal to 1, and $\langle \mathbb{1}, ze_8, ze_8 \rangle = -\frac{1}{2}$.

By the concavity axiom, $\langle e_5, e_{14}, e_{14} \rangle$, $\langle e_6, e_{13}, e_{14} \rangle$, $\langle e_9, e_{10}, e_{14} \rangle$, and $\langle e_9, e_{11}, e_{13} \rangle$ are all equal to 1.

By the index-zero axiom, $\langle e_7, e_{13}, e_{13} \rangle$, $\langle e_{10}, e_{10}, e_{13} \rangle$, and $\langle e_{11}, e_{11}, e_{11} \rangle$ are all equal to -2 .

The correlator $\langle e_8, e_{11}, xe_{14} \rangle$ may be non-zero, and the composition axiom can be used to compute it. However, using associativity of the product we can compute the ring structure on $\mathcal{H}_{S_{11},G}$ without it.

Consider the map

$$\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{11},G}$$

defined by $X \mapsto e_9$, $Y \mapsto e_{13}$, and $Z \mapsto e_{14}$ extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_{13}^2 = -2e_9$, $e_{14}^4 = -2e_{13}e_9$, and $e_{13}e_{14} * e_{14}^2 = 0$.

Hence $(2X + Y^2, 2XY + Z^4, YZ^3) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(2X + Y^2, 2XY + Z^4, YZ^3) = \mathcal{Q}_{S_{11}^T}$ has dimension 13, we deduce the inclusion is in fact equality, and $\mathcal{H}_{S_{11},G} \cong \mathcal{Q}_{S_{11}^T}$.

$S_{12} = \mathbf{x}^2\mathbf{z} + \mathbf{y}\mathbf{z}^2 + \mathbf{x}\mathbf{y}^3$.

$$\begin{aligned} \mathcal{J} &= (2xz, z^2 + 3xy^2, x^2 + 2yz) & q_x &= \frac{4}{13}, q_y = \frac{3}{13}, q_z = \frac{5}{13}, \hat{c} = \frac{15}{13} & G &= \langle J \rangle = \mathbb{Z}/13\mathbb{Z} \\ \text{Fix } J^k &= \begin{cases} \mathbb{C}_{xyz}^3 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} & \mathcal{Q}_{|\text{Fix } J^k} &= \begin{cases} \langle 1, y, x, z, y^2, xy, yz, xz, xy^2, y^2z, xyz, xy^2z \rangle, \mu = 12 \\ \langle 1 \rangle \end{cases} \end{aligned}$$

k	1	2	3	4	5	6	7	8	9	10	11	12
$ G \cdot \deg_W$	0	24	22	20	18	16	14	12	10	8	6	30
invariants	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbb{1}, \mathbb{1}, e_{12} \rangle$, $\langle \mathbb{1}, e_2, e_{11} \rangle$, $\langle \mathbb{1}, e_3, e_{10} \rangle$, $\langle \mathbb{1}, e_4, e_9 \rangle$, $\langle \mathbb{1}, e_5, e_8 \rangle$, and $\langle \mathbb{1}, e_6, e_7 \rangle$ are all equal to 1.

By the concavity axiom, $\langle e_5, e_{11}, e_{11} \rangle$, $\langle e_6, e_{10}, e_{11} \rangle$, $\langle e_7, e_9, e_{11} \rangle$, and $\langle e_8, e_9, e_{10} \rangle$ are all equal to 1.

By the index-zero axiom, $\langle e_7, e_{10}, e_{10} \rangle = -2$, $\langle e_8, e_{11}, e_{11} \rangle = -2$, and $\langle e_9, e_9, e_9 \rangle = -3$.

Consider the map

$$\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{12},G}$$

defined by $X \mapsto e_9$, $Y \mapsto e_{10}$, and $Z \mapsto e_{11}$ extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_9^2 = -3e_4 = -3e_{10}e_{11}^2$, $e_{10}^2 = -2e_6 = -2e_9e_{11}$, and $e_{11}^3 = -2e_5 = -2e_9e_{10}$.

Hence $(Y^2 + 2XZ, Z^3 + 2XY, X^2 + 3YZ^2) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(Y^2 + 2XZ, Z^3 + 2XY, X^2 + 3YZ^2) = \mathcal{Q}_{S_{12}^T}$ has dimension 13, we deduce the inclusion is in fact equality, and $\mathcal{H}_{S_{12},G} \cong \mathcal{Q}_{S_{12}^T}$.

$U_{12} = \mathbf{x}^3 + \mathbf{y}^3 + \mathbf{z}^4$. By Axiom 8, the Mirror Symmetry Conjecture holds here because it holds for the simple singularities $A_2 = x^3$ and $A_3 = z^4$.

$Z_{11} = \mathbf{x}^3\mathbf{y} + \mathbf{y}^5$.

$$\begin{aligned} \mathcal{J} &= (3x^2y, x^3 + 5y^4) & q_x &= \frac{4}{15}, q_y = \frac{3}{15}, \hat{c} = \frac{16}{15} & G &= \langle J \rangle = \mathbb{Z}/15\mathbb{Z} \\ \text{Fix } J^k &= \begin{cases} \mathbb{C}_{xy}^2 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } k = 5, 10 \\ 0 & \text{otherwise} \end{cases} & \mathcal{Q}_{|\text{Fix } J^k} &= \begin{cases} \langle 1, y, x, y^2, xy, x^2, y^3, xy^2, x^3, xy^3, x^4 \rangle, \mu = 11 \\ \langle 1, y, y^2, y^3 \rangle, \mu = 4 \\ \langle 1 \rangle \end{cases} \end{aligned}$$

k	0	1	2	3	4	6	7	8	9	11	12	13	14
$ G \cdot \deg_W$	16	0	14	28	12	10	24	8	22	20	4	18	32
invariants	$x^2 e_0$	$\mathbb{1}$	e_2	e_3	e_4	e_6	e_7	e_8	e_9	e_{11}	e_{12}	e_{13}	e_{14}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbb{1}, \mathbb{1}, e_{14} \rangle$, $\langle \mathbb{1}, e_2, e_{13} \rangle$, $\langle \mathbb{1}, e_3, e_{12} \rangle$, $\langle \mathbb{1}, e_4, e_{11} \rangle$, $\langle \mathbb{1}, e_6, e_9 \rangle$, and $\langle \mathbb{1}, e_7, e_8 \rangle$ are all equal to 1, and $\langle \mathbb{1}, x^2 e_0, x^2 e_0 \rangle = -\frac{1}{3}$.

By the concavity axiom, $\langle e_2, e_2, e_{12} \rangle$, $\langle e_2, e_6, e_8 \rangle$, $\langle e_4, e_6, e_6 \rangle$, $\langle e_6, e_{12}, e_{13} \rangle$, $\langle e_7, e_{12}, e_{12} \rangle$, and $\langle e_8, e_{11}, e_{12} \rangle$ are all equal to 1.

By the index-zero axiom, $\langle e_4, e_4, e_8 \rangle = -3$.

The composition axiom can be used to compute $\langle x^2 e_0, e_4, e_{12} \rangle$, and $\langle x^2 e_0, e_8, e_8 \rangle$, but we do not need these to establish the desired isomorphism.

Consider the map

$$\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{11}, G}$$

defined by $X \mapsto e_6$, and $Y \mapsto e_{12}$ extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_{12}^3 * e_{12}^2 = -3e_{11} = -3e_6^2$ and $e_6 e_{12}^4 = 0$.

Hence $(3X^2 + Y^5, 5XY^4) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2 + Y^5, 5XY^4) = \mathcal{Q}_{Z_{11}^T}$ has dimension 13, we deduce the inclusion is in fact equality, and $\mathcal{H}_{Z_{11}, G} \cong \mathcal{Q}_{Z_{11}^T}$.

$$\mathbf{Z}_{12} = \mathbf{x}^3 \mathbf{y} + \mathbf{x} \mathbf{y}^4.$$

$$\mathcal{J} = (3x^2y + y^4, x^3 + 4xy^3) \quad q_x = \frac{3}{11}, q_y = \frac{2}{11}, \hat{c} = \frac{12}{11} \quad G = \langle J \rangle = \mathbb{Z}/11\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}_{xy}^2 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, y, x, y^2, xy, y^3, x^2, xy^2, x^2y, xy^3, x^2y^2, x^2y^3 \rangle, \mu = 12 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	4	5	6	7	8	9	10
$ G \cdot \deg_W$	12	0	10	20	8	18	6	16	4	14	24
invariants	$x^2 e_0, y^3 e_0$	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbb{1}, \mathbb{1}, e_{10} \rangle$, $\langle \mathbb{1}, e_2, e_9 \rangle$, $\langle \mathbb{1}, e_3, e_8 \rangle$, $\langle \mathbb{1}, e_4, e_7 \rangle$, and $\langle \mathbb{1}, e_5, e_6 \rangle$, $\langle \mathbb{1}, e_9, e_9 \rangle$ are all equal to 1.

Also, $\langle \mathbb{1}, x^2 e_0, x^2 e_0 \rangle = \frac{1}{11}$, $\langle \mathbb{1}, x^2 e_0, y^3 e_0 \rangle = -\frac{4}{11}$ and $\langle \mathbb{1}, y^3 e_0, y^3 e_0 \rangle = -\frac{3}{11}$.

By the concavity axiom, $\langle e_2, e_2, e_8 \rangle$, $\langle e_2, e_4, e_6 \rangle$, $\langle e_6, e_8, e_9 \rangle$, and $\langle e_7, e_8, e_8 \rangle$ are all equal to 1.

By the index-zero axiom, $\langle e_4, e_4, e_4 \rangle = -3$.

The correlators $\langle x^2 e_0, e_4, e_8 \rangle$, $\langle y^3 e_0, e_4, e_8 \rangle$, $\langle x^2, e_6, e_6 \rangle$ and $\langle y^3 e_0, e_6, e_6 \rangle$ are also non-zero, and we use the composition axiom to extract the necessary information (although we avoid computing individual correlators directly).

Note

$$e_6^2 = \sum_{\mu, \nu \in \{x^2 e_0, y^3 e_0\}} \langle e_6, e_6, \mu \rangle \eta^{\mu, \nu} \nu$$

$$e_6^3 = \sum_{\mu, \nu \in \{x^2 e_0, y^3 e_0\}} \langle e_6, e_6, \mu \rangle \eta^{\mu, \nu} \langle \nu, e_6, e_6 \rangle e_5,$$

By the composition axiom, the coefficient of e_5 here is just the value of the four pointed class $\Lambda_{0,4}^{Z_{12}}(e_6, e_6, e_6, e_6)$. This four-pointed class has codimension zero, and $l_x = -2$ and $l_y = 0$, so its value is the y -degree of the Witten map, which is -4 .

Consider the map

$$\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{12}, G}$$

defined by $X \mapsto e_6$, and $Y \mapsto e_8$ extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this maps onto each twisted sector.

To prove surjectivity, we need to check that it also maps onto the untwisted sector. This is equivalent to the linear independence of e_6^2 and e_8^3 . We note the following identities, which are direct consequences of the three-point correlator values cited above:

$$e_6^2 * e_6 = -4e_5$$

$$e_8^3 * e_6 = e_8^2 * (e_8 e_6) = e_4 * e_2 = e_5$$

$$e_6^2 * e_8 = e_6 * (e_6 e_8) = e_6 * e_2 = e_7$$

$$e_8^3 * e_8 = e_8^2 * e_8^2 = e_4 * e_4 = -3e_7.$$

Putting $\mu = e_6^2 + 4e_8^3$ and $\nu = 3e_6 + e_8^3$, we see that

$$\mu * e_6 = 0$$

$$\mu * e_8 = -11e_7$$

$$\nu * e_6 = -11e_5$$

$$\nu * e_8 = 0,$$

and conclude that μ and ν are linearly independent combinations of e_6^2 and e_8^3 , yielding surjectivity of the map φ .

The above identities also show $(3X^2Y + Y^4, X^3 + 4XY^3) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2Y + Y^4, X^3 + 4XY^3) = \mathcal{Z}_{Z_{12}^T}$ has dimension 12, we deduce the inclusion is in fact equality, and $\mathcal{H}_{Z_{12}, G} \cong \mathcal{Z}_{Z_{12}^T}$.

$$\mathbf{Z}_{13} = \mathbf{x}^3 \mathbf{y} + \mathbf{y}^6.$$

$$\mathcal{J} = (3x^2y, x^3 + 6y^5) \quad q_x = \frac{5}{18}, q_y = \frac{3}{18}, \hat{c} = \frac{20}{18} \quad G = \langle J \rangle = \mathbb{Z}/18\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}_{xy}^2 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } k = 6, 12 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}_{|\text{Fix } J^k} = \begin{cases} \langle 1, y, x, y^2, xy, y^3, x^2, xy^2, y^4, xy^3, y^5, xy^4, xy^5 \rangle, \mu = 13 \\ \langle 1, y, y^2, y^3, y^4 \rangle, \mu = 5 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	4	5	7	8	9	10	11	13	14	15	16	17
$ G \cdot \deg_W$	20	0	16	32	12	28	24	4	20	36	16	12	28	8	24	40
invariants	$x^2 e_0$	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_7	e_8	e_9	e_{10}	e_{11}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbf{1}, \mathbf{1}, e_{17} \rangle$, $\langle \mathbf{1}, e_2, e_{16} \rangle$, $\langle \mathbf{1}, e_3, e_{15} \rangle$, $\langle \mathbf{1}, e_4, e_{14} \rangle$, $\langle \mathbf{1}, e_5, e_{13} \rangle$, $\langle \mathbf{1}, e_7, e_{11} \rangle$, $\langle \mathbf{1}, e_8, e_{10} \rangle$, $\langle \mathbf{1}, e_9, e_9 \rangle$ are all equal to 1, and $\langle \mathbf{1}, x^2 e_0, x^2 e_0 \rangle = -\frac{1}{3}$.

By the concavity axiom, $\langle e_2, e_2, e_{15} \rangle$, $\langle e_2, e_4, e_{13} \rangle$, $\langle e_2, e_8, e_9 \rangle$, $\langle e_3, e_8, e_8 \rangle$, $\langle e_4, e_7, e_8 \rangle$, $\langle e_7, e_{15}, e_{15} \rangle$, $\langle e_8, e_{13}, e_{16} \rangle$, $\langle e_8, e_{14}, e_{15} \rangle$, $\langle e_9, e_{13}, e_{15} \rangle$, and $\langle e_{11}, e_{13}, e_{13} \rangle$ are all equal to 1.

By the index-zero axiom, $\langle e_4, e_4, e_{11} \rangle$ and $\langle e_{11}, e_{11}, e_{15} \rangle$ are both equal to -3 .

The composition axiom can be used to compute $\langle x^2 e_0, e_4, e_{15} \rangle$, and $\langle x^2 e_0, e_8, e_{11} \rangle$, but we do not need these to establish the desired isomorphism.

Consider the map

$$\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{13}, G},$$

defined by $X \mapsto e_{13}$, and $Y \mapsto e_8$ extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_8^3 * e_8^3 = -3e_7 = -3e_{13}^2$ and $e_{13}e_8^5 = 0$.

Hence $(3X^2 + Y^6, 6XY^5) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2 + Y^6, 6XY^5) = \mathcal{Q}_{Z_{13}^T}$ has dimension 16, we deduce the inclusion is in fact equality, and $\mathcal{H}_{Z_{13}, G} \cong \mathcal{Q}_{Z_{13}^T}$.

$\mathbf{W}_{12} = \mathbf{x}^4 + \mathbf{y}^5$. By Axiom 8, the conjecture holds for W_{12} because it holds for the simple singularities $A_3 = x^4$ and $A_4 = y^5$.

$\mathbf{W}_{13} = \mathbf{x}^4 + \mathbf{xy}^4$.

$$\mathcal{J} = (4x^3 + y^4, 4xy^3) \quad q_x = \frac{4}{16}, q_y = \frac{3}{16}, \hat{c} = \frac{18}{16} \quad G = \langle J \rangle = \mathbb{Z}/16\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}_{xy}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } k = 4, 8, 12 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, y^4, y^5, y^6 \rangle, \mu = 13 \\ \langle 1, x, x^2 \rangle, \mu = 3 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	5	6	7	9	10	11	13	14	15
$ G \cdot \deg_W$	18	0	14	28	24	6	20	16	30	12	8	22	36
invariants	$y^3 e_0$	$\mathbf{1}$	e_2	e_3	e_5	e_6	e_7	e_9	e_{10}	e_{11}	e_{13}	e_{14}	e_{15}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbf{1}, \mathbf{1}, e_{15} \rangle$, $\langle \mathbf{1}, e_2, e_{14} \rangle$, $\langle \mathbf{1}, e_3, e_{13} \rangle$, $\langle \mathbf{1}, e_5, e_{11} \rangle$, $\langle \mathbf{1}, e_6, e_{10} \rangle$, and $\langle \mathbf{1}, e_7, e_9 \rangle$ are all equal to 1, and $\langle \mathbf{1}, y^3 e_0, y^3 e_0 \rangle = -\frac{1}{4}$.

By the concavity axiom, $\langle e_2, e_2, e_{13} \rangle$, $\langle e_2, e_6, e_9 \rangle$, $\langle e_5, e_6, e_6 \rangle$, $\langle e_6, e_{13}, e_{14} \rangle$, $\langle e_7, e_{13}, e_{13} \rangle$, and $\langle e_9, e_{11}, e_{13} \rangle$ are all equal to 1.

By the index-zero axiom, $\langle e_{11}, e_{11}, e_{11} \rangle = -4$.

The correlator $\langle y^3 e_0, e_6, e_{11} \rangle$ may be non-zero, and the composition axiom can be used to compute it. However, using associativity of the product we can compute the ring structure on $\mathcal{H}_{W_{13}, G}$ without it.

Consider the map

$$\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{W_{13}, G}$$

defined by $X \mapsto e_6$, and $Y \mapsto e_{13}$, extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_6^2 * e_6^2 = -4e_5 = -4e_{13}^3$, and $e_6^2 * (e_6 e_{13}) = 0$.

Hence $(4X^3Y, X^4 + 4Y^3) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y]/(4X^3Y, X^4 + 4Y^3) = \mathcal{Q}_{W_{13}^T}$ has dimension 13, we deduce the inclusion is in fact equality, and $\mathcal{H}_{W_{13}, G} \cong \mathcal{Q}_{W_{13}^T}$.

$E_{12} = \mathbf{x}^3 + \mathbf{y}^7$. By Axiom 8, the Mirror Symmetry Conjecture holds for E_{12} because it holds for the simple singularities $A_2 = x^3$ and $A_6 = y^7$.

$E_{13} = \mathbf{x}^3 + \mathbf{xy}^5$.

$$\mathcal{J} = (4x^2 + y^5, 5xy^4) \quad q_x = \frac{5}{15}, q_y = \frac{2}{15}, \hat{c} = \frac{16}{15} \quad G = \langle J \rangle = \mathbb{Z}/15\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}_{xy}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } k = 3, 6, 9, 12 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, y, y^2, x, y^3, xy, y^4, xy^2, y^5, xy^3, y^6, y^7, y^8 \rangle, \mu = 13 \\ \langle 1, x \rangle, \mu = 2 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	4	5	7	8	10	11	13	14
$ G \cdot \deg_W$	16	0	14	12	26	24	8	6	20	18	32
invariants	$y^4 e_0$	$\mathbb{1}$	e_2	e_4	e_5	e_7	e_8	e_{10}	e_{11}	e_{13}	e_{14}

Potential non-zero correlators:

By the pairing axiom, $\langle \mathbb{1}, \mathbb{1}, e_{14} \rangle$, $\langle \mathbb{1}, e_2, e_{13} \rangle$, $\langle \mathbb{1}, e_4, e_{11} \rangle$, $\langle \mathbb{1}, e_5, e_{10} \rangle$, and $\langle \mathbb{1}, e_7, e_8 \rangle$ are all equal to 1, and $\langle \mathbb{1}, y^4 e_0, y^4 e_0 \rangle = -\frac{1}{5}$.

By the concavity axiom, $\langle e_2, e_4, e_{10} \rangle$, $\langle e_4, e_4, e_8 \rangle$, $\langle e_8, e_{10}, e_{13} \rangle$, and $\langle e_{10}, e_{10}, e_{11} \rangle$ are all equal to 1.

By the composition axiom, we have

$$-5 = \langle e_8, e_8, y^4 e_0 \rangle \eta^{y^4 e_0, y^4 e_0} \langle y^4 e_0, e_8, e_8 \rangle,$$

so $\langle e_8, e_8, y^4 e_0 \rangle = \pm 1$.

Consider the map

$$\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{E_{13}, G}$$

defined by $X \mapsto e_8$, and $Y \mapsto e_{10}$, extending by \mathbb{C} -linearity and multiplicativity. One can check directly that this map is surjective.

Our correlators tell us that $e_8 * e_8 e_{10} = 0$ and $e_8^2 * e_8 = -5e_7 = -5e_{10}^4$.

Hence $(3X^2Y, X^3 + 5Y^4) \subseteq \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2Y, X^3 + 5Y^4) = \mathcal{Q}_{E_{13}^T}$ has dimension 11, we deduce the inclusion is in fact equality, and $\mathcal{H}_{E_{13}, G} \cong \mathcal{Q}_{E_{13}^T}$.

$E_{14} = \mathbf{x}^3 + \mathbf{y}^8$. By Axiom 8, the Mirror Symmetry Conjecture holds for E_{14} because it holds for the simple singularities $A_2 = x^3$ and $A_7 = y^8$.

2.2. Bimodal Singularities. We now turn to the fourteen exceptional bimodal families listed in Arnol'd [1]. We note that the conjecture has already been shown for E_{18} , E_{20} , U_{16} , W_{18} , Q_{16} , and Q_{18} in [4] as these are sums of simple singularities. Also E_{19} was constructed in the introduction, so in this section, we will consider the eight remaining families, together with their mirror partners.

In [4], the construction of the FJRW-ring does not include singularities with a weight greater than or equal to $1/2$. In particular, it has not been verified that the required compact moduli space exists in these cases, and that the three-point correlators satisfy the proper axioms. However, Fan-Jarvis-Ruan have proven that the construction works for two particular singularities having a variable with weight $1/2$, namely A_2 and D_n^T for n even, and it is expected to work for all cases with weight $\frac{1}{2}$ (see [3]). When our singularities have variables of weight $\frac{1}{2}$, we will treat them assuming the FJR axioms hold.

In this paper, we have included four examples of singularities with this property, namely Q_{17}^T , S_{17}^T , $Q_{2,0}^T$ and $S_{1,0}^T$. In each case, we have shown how the isomorphism between the FJRW-rings and the Milnor rings should work under the assumption that a certain three-point correlator is non-zero.

$$\mathbf{Z}_{17} = \mathbf{x}^3\mathbf{y} + \mathbf{y}^8.$$

$$\mathcal{J} = (3x^2y, x^3 + 8y^7) \quad q_x = \frac{7}{24}, \quad q_y = \frac{3}{24}, \quad \hat{c} = \frac{28}{24} \quad G = \langle J \rangle \cong \mathbb{Z}/24\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } 8|k, k \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, x, x^2, y, \dots, y^7, xy, \dots, xy^7 \rangle, \mu = 17 \\ \langle 1, y, \dots, y^6 \rangle, \mu = 7 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	4	5	6	7	9	10	11
$ G \cdot \deg_W$	28	0	20	40	12	32	52	24	16	36	8
invariants	x^2e_0	$\mathbf{1}$	e_2	e_3	e_4	e_5	e_6	e_7	e_9	e_{10}	e_{11}

k	12	13	14	15	17	18	19	20	21	22	23
$ G \cdot \deg_W$	28	48	20	12	32	4	24	44	16	36	56
invariants	e_{12}	e_{13}	e_{14}	e_{15}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbf{1}, \mathbf{1}, e_{23} \rangle$, $\langle \mathbf{1}, e_2, e_{22} \rangle$, $\langle \mathbf{1}, e_3, e_{21} \rangle$, $\langle \mathbf{1}, e_4, e_{20} \rangle$, $\langle \mathbf{1}, e_5, e_{19} \rangle$, $\langle \mathbf{1}, e_6, e_{18} \rangle$, $\langle \mathbf{1}, e_7, e_{17} \rangle$, $\langle e_1, e_9, e_{15} \rangle$, $\langle \mathbf{1}, e_{10}, e_{14} \rangle$, $\langle \mathbf{1}, e_{11}, e_{13} \rangle$ and $\langle \mathbf{1}, e_{12}, e_{12} \rangle$ are equal to 1, and $\langle x^2e_0, x^2e_0, \mathbf{1} \rangle = -1/3$.

By the Concavity axiom $\langle e_2, e_2, e_{21} \rangle$, $\langle e_2, e_4, e_{19} \rangle$, $\langle e_2, e_5, e_{18} \rangle$, $\langle e_2, e_9, e_{14} \rangle$, $\langle e_2, e_{11}, e_{12} \rangle$, $\langle e_3, e_4, e_{18} \rangle$, $\langle e_3, e_{11}, e_{11} \rangle$, $\langle e_4, e_4, e_{17} \rangle$, $\langle e_4, e_9, e_{12} \rangle$, $\langle e_4, e_{10}, e_{11} \rangle$, $\langle e_5, e_9, e_{11} \rangle$, $\langle e_7, e_9, e_9 \rangle$, $\langle e_9, e_{18}, e_{22} \rangle$, $\langle e_9, e_{19}, e_{21} \rangle$, $\langle e_{10}, e_{18}, e_{21} \rangle$, $\langle e_{11}, e_{17}, e_{21} \rangle$, $\langle e_{11}, e_{18}, e_{20} \rangle$, $\langle e_{11}, e_{19}, e_{19} \rangle$, $\langle e_{12}, e_{18}, e_{19} \rangle$, $\langle e_{13}, e_{18}, e_{18} \rangle$, and $\langle e_{14}, e_{17}, e_{18} \rangle$ are all equal to 1.

By the Index Zero axiom $\langle e_4, e_7, e_{14} \rangle = \langle e_7, e_7, e_{11} \rangle = \langle e_7, e_{21}, e_{21} \rangle = \langle e_{14}, e_{14}, e_{21} \rangle = -3$.

By the composition axiom $\langle x^2e_0, e_4, e_{21} \rangle = \langle x^2e_0, e_7, e_{18} \rangle = \langle x^2e_0, e_{11}, e_{14} \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{17}, G}$ defined by $X \mapsto e_9$ and $Y \mapsto e_{18}$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_9^2 = e_{17}$, $e_{18}^8 = -3e_{17}$, and $e_9e_{18}^7 = 0$.

Hence $(3X^2 + Y^8, 8XY^8) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2 + Y^8, 8XY^8) = \mathcal{Q}_{Z_{17}^T}$ has dimension 22, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Z_{17}, G} \cong \mathcal{Q}_{Z_{17}^T}$.

$$\mathbf{Z}_{17}^T = \mathbf{x}^3 + \mathbf{xy}^8.$$

$$\mathcal{J} = (3x^2 + y^8, 8xy^7) \quad q_x = \frac{8}{24}, q_y = \frac{2}{24}, \hat{c} = \frac{28}{24} \quad G \cong \mathbb{Z}/24\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/12\mathbb{Z}$$

Since J does not generate the maximal group of symmetries, we will use the generator $g = (\zeta^{16}, \zeta)$, where $\zeta^{24} = 1$. We will index our graded FJRW-ring by powers of g .

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } 3|k, k \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } g^k} = \begin{cases} \langle 1, x, x^2, y, \dots, y^7, xy, \dots, xy^6, x^2y, \dots, x^2y^6 \rangle, \mu = 22 \\ \langle 1, x \rangle, \mu = 2 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23
$ G \cdot \deg_W$	28	14	0	20	6	26	12	32	18	38	24	44	30	50	36	56	42
invariants	y^7e_0	e_1	$\mathbb{1}$	e_4	e_5	e_7	e_8	e_{10}	e_{11}	e_{13}	e_{14}	e_{16}	e_{17}	e_{19}	e_{20}	e_{22}	e_{23}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, e_1, e_{23} \rangle, \langle \mathbb{1}, \mathbb{1}, e_{22} \rangle, \langle \mathbb{1}, e_4, e_{20} \rangle, \langle \mathbb{1}, e_5, e_{19} \rangle, \langle \mathbb{1}, e_7, e_{17} \rangle, \langle \mathbb{1}, e_8, e_{16} \rangle, \langle \mathbb{1}, e_{10}, e_{14} \rangle$, and $\langle \mathbb{1}, e_{11}, e_{13} \rangle$ are equal to 1, and $\langle y^7e_0, y^7e_0, \mathbb{1} \rangle = -1/8$.

By the Concavity axiom $\langle e_1, e_5, e_{20} \rangle, \langle e_1, e_8, e_{17} \rangle, \langle e_1, e_{11}, e_{14} \rangle, \langle e_4, e_5, e_{17} \rangle, \langle e_4, e_8, e_{14} \rangle, \langle e_4, e_{11}, e_{11} \rangle, \langle e_5, e_5, e_{16} \rangle, \langle e_5, e_7, e_{14} \rangle, \langle e_5, e_8, e_{13} \rangle, \langle e_5, e_{10}, e_{11} \rangle, \langle e_7, e_8, e_{11} \rangle$, and $\langle e_8, e_8, e_{10} \rangle$ are all equal to 1.

By the composition axiom $\langle e_1, e_1, y^7e_0 \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{17}, G}$ defined by $X \mapsto e_1$ and $Y \mapsto e_5$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_1^3 = -8e_{23}$, $e_5^7 = e_{23}$, and $e_1^2e_5 = 0$.

Hence $(3X^2Y, X^3 + 8Y^7) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2Y, X^3 + 8Y^7) = \mathcal{Q}_{Z_{17}}$ has dimension 17, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Z_{17}, G} \cong \mathcal{Q}_{Z_{17}}$.

$$\mathbf{Z}_{18} = \mathbf{x}^3\mathbf{y} + \mathbf{xy}^6.$$

$$\mathcal{J} = (3x^2y + y^6, x^3 + 6xy^5) \quad q_x = \frac{5}{17}, q_y = \frac{2}{17}, \hat{c} = \frac{20}{17} \quad G = \langle J \rangle \cong \mathbb{Z}/17\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^0} = \begin{cases} \langle 1, x, x^2, y, \dots, y^7, xy, \dots, xy^5, x^2y, \dots, x^2y^6 \rangle, \mu = 18 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$ G \cdot \deg_W$	20	0	14	28	8	22	36	16	30	10	24	4	18	32	12	26	40
invariants	x^2e_0, y^5e_0	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{16} \rangle, \langle \mathbb{1}, e_2, e_{15} \rangle, \langle \mathbb{1}, e_3, e_{14} \rangle, \langle \mathbb{1}, e_4, e_{13} \rangle, \langle \mathbb{1}, e_5, e_{12} \rangle, \langle \mathbb{1}, e_6, e_{11} \rangle, \langle \mathbb{1}, e_7, e_{10} \rangle$, and $\langle \mathbb{1}, e_8, e_9 \rangle$ are equal to 1, and $\langle x^2e_0, x^2e_0, \mathbb{1} \rangle = -6/17$, $\langle x^2e_0, y^5e_0, \mathbb{1} \rangle = 1/17$, $\langle y^5e_0, y^5e_0, \mathbb{1} \rangle = -3/17$.

By the Concavity axiom $\langle e_2, e_2, e_{14} \rangle, \langle e_2, e_4, e_{12} \rangle, \langle e_2, e_5, e_{11} \rangle, \langle e_2, e_7, e_9 \rangle, \langle e_3, e_4, e_{11} \rangle, \langle e_4, e_4, e_{10} \rangle, \langle e_4, e_5, e_9 \rangle, \langle e_9, e_{11}, e_{15} \rangle, \langle e_9, e_{12}, e_{14} \rangle, \langle e_{10}, e_{11}, e_{14} \rangle, \langle e_{11}, e_{11}, e_{13} \rangle$, and $\langle e_{11}, e_{12}, e_{12} \rangle$ are all equal to 1.

By the Index Zero axiom $\langle e_4, e_7, e_7 \rangle = \langle e_7, e_{14}, e_{14} \rangle = -3$

The remaining potentially non-zero three point correlators cannot be determined from the axioms alone. These correlators are $\langle x^2e_0, e_4, e_{14} \rangle, \langle y^5e_0, e_4, e_{14} \rangle, \langle x^2e_0, e_7, e_{11} \rangle, \langle y^5e_0, e_7, e_{11} \rangle, \langle x^2e_0, e_9, e_9 \rangle$, and $\langle y^5e_0, e_9, e_9 \rangle$.

In order to prove this we consider the homomorphism $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{18}, G}$ defined by $X \mapsto e_9$ and $Y \mapsto e_{11}$. It takes some work to show that φ is surjective in this case. One can check that φ maps *onto* the one-dimensional sectors.

However, the sector corresponding to the identity in G is two-dimensional. To prove surjectivity, it suffices to show that e_{11}^5 and e_9^2 are linearly independent.

We note the following identities, which are direct consequences of the three-point correlator values cited above:

$$\begin{aligned} e_9^2 * e_9 &= -6e_8 \\ e_9^2 * e_{11} &= e_{10} \end{aligned}$$

$$\begin{aligned} e_{11}^5 * e_9 &= e_8 \\ e_{11}^5 * e_{11} &= -3e_{10}. \end{aligned}$$

Putting $\mu = e_9^2 + 6e_{11}^5$ and $\nu = 3e_9^2 + e_{11}^5$, we see that

$$\begin{aligned} \mu * e_9 &= 0 \\ \mu * e_{11} &= -17e_{10} \end{aligned}$$

$$\begin{aligned} \nu * e_9 &= -17e_8 \\ \nu * e_{11} &= 0, \end{aligned}$$

and conclude that μ and ν are linearly independent combinations of e_9^2 and e_{11}^5 , yielding surjectivity of the map φ .

The above identities also tell us that $(3X^2Y + Y^6, X^3 + 6XY^5) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2Y + Y^6, X^3 + 6XY^5) = \mathcal{Q}_{Z_{18}}$ has dimension 18 we deduce that we have an isomorphism $\mathcal{Q}_{Z_{18}} \cong \mathcal{H}_{Z_{18}, G}$.

$\mathbf{Z}_{19} = \mathbf{x}^3\mathbf{y} + \mathbf{y}^9$.

$$\begin{aligned} \mathcal{J} &= (3x^2y, x^3 + 9y^8) & q_x &= \frac{8}{27}, q_y = \frac{3}{27}, \hat{c} = \frac{32}{27} & G &= \langle J \rangle \cong \mathbb{Z}/27\mathbb{Z} \\ \text{Fix } J^k &= \begin{cases} \mathbb{C}_{xy}^2 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } 9|k, k \neq 0 \\ 0 & \text{otherwise} \end{cases} & \mathcal{Q}|_{\text{Fix } J^k} &= \begin{cases} \langle 1, x, x^2, y, \dots, y^8, xy, xy^2, \dots, xy^8 \rangle, \mu = 19 \\ \langle 1, y, y^2, y^3, y^4, y^5, y^6, y^7 \rangle, \mu = 8 \\ \langle 1 \rangle \end{cases} \end{aligned}$$

k	0	1	2	3	4	5	6	7	8	10	11	12	13	14	15	16	17
$ G \cdot \deg_W$	32	0	22	44	12	34	56	24	46	36	4	26	48	16	38	60	28
invariants	x^2e_0	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}

k	19	20	21	22	23	24	25	26
$ G \cdot \deg_W$	18	40	8	30	52	20	42	64
invariants	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{26} \rangle, \langle \mathbb{1}, e_2, e_{25} \rangle, \langle \mathbb{1}, e_3, e_{24} \rangle, \langle \mathbb{1}, e_4, e_{23} \rangle, \langle \mathbb{1}, e_5, e_{22} \rangle, \langle \mathbb{1}, e_6, e_{21} \rangle, \langle \mathbb{1}, e_7, e_{20} \rangle, \langle \mathbb{1}, e_8, e_{19} \rangle, \langle \mathbb{1}, e_{10}, e_{17} \rangle, \langle \mathbb{1}, e_{11}, e_{16} \rangle, \langle \mathbb{1}, e_{12}, e_{15} \rangle, \text{ and } \langle \mathbb{1}, e_{13}, e_{14} \rangle$ are equal to 1, and $\langle x^2e_0, x^2e_0, \mathbb{1} \rangle = -1/3$,

By the Concavity axiom $\langle e_2, e_2, e_{24} \rangle, \langle e_2, e_4, e_{22} \rangle, \langle e_2, e_5, e_{21} \rangle, \langle e_2, e_7, e_{19} \rangle, \langle e_2, e_{11}, e_{15} \rangle, \langle e_2, e_{12}, e_{14} \rangle, \langle e_3, e_4, e_{21} \rangle, \langle e_3, e_{11}, e_{14} \rangle, \langle e_4, e_4, e_{20} \rangle, \langle e_4, e_5, e_{19} \rangle, \langle e_4, e_{10}, e_{14} \rangle, \langle e_4, e_{11}, e_{13} \rangle, \langle e_4, e_{12}, e_{12} \rangle, \langle e_5, e_{11}, e_{12} \rangle, \langle e_6, e_{11}, e_{11} \rangle,$

$\langle e_7, e_{10}, e_{11} \rangle, \langle e_{10}, e_{21}, e_{24} \rangle, \langle e_{11}, e_{19}, e_{25} \rangle, \langle e_{11}, e_{20}, e_{24} \rangle, \langle e_{11}, e_{21}, e_{23} \rangle, \langle e_{11}, e_{22}, e_{22} \rangle, \langle e_{12}, e_{19}, e_{24} \rangle, \langle e_{12}, e_{21}, e_{22} \rangle, \langle e_{13}, e_{21}, e_{21} \rangle, \langle e_{14}, e_{17}, e_{24} \rangle, \langle e_{14}, e_{19}, e_{22} \rangle, \langle e_{14}, e_{20}, e_{21} \rangle, \langle e_{15}, e_{19}, e_{21} \rangle$, and $\langle e_{17}, e_{19}, e_{19} \rangle$ are all equal to 1.

By the Index Zero axiom $\langle e_4, e_7, e_{17} \rangle = \langle e_7, e_7, e_{14} \rangle = \langle e_7, e_{24}, e_{24} \rangle = \langle e_{17}, e_{17}, e_{21} \rangle = -3$

By the composition axiom $\langle x^2 e_0, e_4, e_{24} \rangle = \langle x^2 e_0, e_7, e_{21} \rangle = \langle x^2 e_0, e_{11}, e_{17} \rangle = \langle x^2 e_0, e_{14}, e_{14} \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{19}, G}$ defined by $X \mapsto e_{19}$ and $Y \mapsto e_{11}$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{19}^2 = e_{10}$, $e_{11}^9 = -3e_{10}$ and $e_{19}e_{11}^8 = 0$.

Hence $(3X^2 + Y^9, 9XY^8) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2 + Y^9, 9XY^8) = \mathcal{Q}_{Z_{19}^T}$ has dimension 25, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Z_{19}, G} \cong \mathcal{Q}_{Z_{19}^T}$.

$$\mathbf{Z}_{19}^T = \mathbf{x}^3 + \mathbf{xy}^9.$$

$$\begin{aligned} \mathcal{J} &= (3x^2 + y^9, 9xy^8) & q_x &= \frac{9}{27}, q_y = \frac{2}{27}, \hat{c} = \frac{32}{27} & G &= \langle J \rangle \cong \mathbb{Z}/27\mathbb{Z} \\ \text{Fix } J^k &= \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } 3|k \\ 0 & \text{otherwise} \end{cases} & \mathcal{Q}|_{\text{Fix } J^k} &= \begin{cases} \langle 1, x, y, \dots, y^8, xy, \dots, xy^7, x^2y, \dots, x^2y^7 \rangle, \mu = 25 \\ \langle 1, x \rangle, \mu = 2 \\ \langle 1 \rangle \end{cases} \end{aligned}$$

k	0	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26
$ G \cdot \deg_W$	32	0	22	12	34	24	46	36	58	48	16	6	28	18	40	30	52	42	64
invariants	$y^8 e_0$	$\mathbb{1}$	e_2	e_4	e_5	e_7	e_8	e_{10}	e_{11}	e_{13}	e_{14}	e_{16}	e_{17}	e_{19}	e_{20}	e_{22}	e_{23}	e_{25}	e_{26}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{26} \rangle, \langle \mathbb{1}, e_2, e_{25} \rangle, \langle \mathbb{1}, e_4, e_{23} \rangle, \langle \mathbb{1}, e_5, e_{22} \rangle, \langle \mathbb{1}, e_7, e_{20} \rangle, \langle \mathbb{1}, e_8, e_{19} \rangle, \langle \mathbb{1}, e_{10}, e_{17} \rangle, \langle \mathbb{1}, e_{11}, e_{16} \rangle$, and $\langle \mathbb{1}, e_{13}, e_{14} \rangle$ are equal to 1, and $\langle y^8 e_0, y^8 e_0, \mathbb{1} \rangle = -1/9$,

By the Concavity axiom $\langle e_2, e_4, e_{22} \rangle, \langle e_2, e_7, e_{19} \rangle, \langle e_2, e_{10}, e_{16} \rangle, \langle e_4, e_4, e_{20} \rangle, \langle e_4, e_5, e_{19} \rangle, \langle e_4, e_7, e_{17} \rangle, \langle e_4, e_8, e_{16} \rangle, \langle e_4, e_{10}, e_{14} \rangle, \langle e_5, e_7, e_{16} \rangle, \langle e_7, e_7, e_{14} \rangle, \langle e_{14}, e_{16}, e_{25} \rangle, \langle e_{14}, e_{19}, e_{22} \rangle, \langle e_{16}, e_{16}, e_{23} \rangle, \langle e_{16}, e_{17}, e_{22} \rangle, \langle e_{16}, e_{19}, e_{20} \rangle$, and $\langle e_{17}, e_{19}, e_{19} \rangle$ are all equal to 1.

By the composition axiom $\langle y^8 e_0, e_4, e_{24} \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{Z_{19}^T, G}$ defined by $X \mapsto e_{14}$ and $Y \mapsto e_{16}$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{14}^3 = -9e_{13}$, $e_{16}^8 = e_{13}$ and $e_{14}^2 e_{16} = 0$.

Hence $(3X^2Y, X^3 + 9Y^8) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2Y, X^3 + 9Y^8) = \mathcal{Q}_{Z_{19}}$ has dimension 19, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Z_{19}^T, G} \cong \mathcal{Q}_{Z_{19}}$.

$$\mathbf{W}_{17} = \mathbf{x}^4 + \mathbf{xy}^5.$$

$$\begin{aligned} \mathcal{J} &= (4x^3 + y^5, 5xy^4) & q_x &= \frac{5}{20}, q_y = \frac{3}{20}, \hat{c} = \frac{24}{20} & G &= \langle J \rangle \cong \mathbb{Z}/20\mathbb{Z} \\ \text{Fix } J^k &= \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } 4|k \\ 0 & \text{otherwise} \end{cases} & \mathcal{Q}|_{\text{Fix } J^k} &= \begin{cases} \langle 1, x, x^2, y, \dots, y^8, xy, xy^2, xy^3, x^2y, x^2y^2, x^2y^3 \rangle, \mu = 17 \\ \langle 1, x, x^2 \rangle, \mu = 3 \\ \langle 1 \rangle \end{cases} \end{aligned}$$

k	0	1	2	3	5	6	7	9	10	11	13	14	15	17	18	19
$ G \cdot \deg_W$	24	0	16	32	24	40	16	8	24	40	32	8	24	16	32	48
invariants	$y^4 e_0$	$\mathbb{1}$	e_2	e_3	e_5	e_6	e_7	e_9	e_{10}	e_{11}	e_{13}	e_{14}	e_{15}	e_{17}	e_{18}	e_{19}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{19} \rangle, \langle \mathbb{1}, e_2, e_{18} \rangle, \langle \mathbb{1}, e_3, e_{17} \rangle, \langle \mathbb{1}, e_5, e_{15} \rangle, \langle \mathbb{1}, e_6, e_{14} \rangle, \langle \mathbb{1}, e_7, e_{13} \rangle, \langle \mathbb{1}, e_9, e_{11} \rangle$, and $\langle \mathbb{1}, e_{10}, e_{10} \rangle$ are all equal to 1, and $\langle y^4 e_0, y^4 e_0, \mathbb{1} \rangle = -1/5$

By the concavity axiom $\langle e_2, e_2, e_{17} \rangle, \langle e_2, e_5, e_{14} \rangle, \langle e_2, e_9, e_{10} \rangle, \langle e_3, e_9, e_9 \rangle, \langle e_5, e_7, e_9 \rangle, \langle e_7, e_{17}, e_{17} \rangle, \langle e_9, e_{14}, e_{18} \rangle, \langle e_9, e_{15}, e_{17} \rangle, \langle e_{10}, e_{14}, e_{17} \rangle, \langle e_{13}, e_{14}, e_{14} \rangle$, are all equal to 1.

By the Index Zero axiom $\langle e_7, e_7, e_7 \rangle = -5$.

By the Composition axiom $\langle y^4 e_0, e_7, e_{14} \rangle = \pm 1$

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{W_{17}, G}$ defined by $X \mapsto e_{14}$ and $Y \mapsto e_9$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{14}^3 e_9 = 0$, $e_{14}^4 = -5e_{13}$, and $e_9^4 = e_{13}$.

Hence $(4X^3 Y, X^4 + 5Y^4) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(4X^3 Y, X^4 + 5Y^4) = \mathcal{Q}_{W_{17}^T}$ has dimension 16, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{W_{17}, G} \cong \mathcal{Q}_{W_{17}^T}$.

$$\mathbf{W}_{17}^T = \mathbf{x}^4 \mathbf{y} + \mathbf{y}^5.$$

$$\mathcal{J} = (4x^3 y, x^4 + 5y^4) \quad q_x = \frac{4}{20}, q_y = \frac{4}{20}, \hat{c} = \frac{24}{20} \quad G \cong \mathbb{Z}/20\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/5\mathbb{Z}$$

Since J does not generate G we will use the generator $g = (\zeta, \zeta^{-4})$, where ζ is a primitive 20-th root of unity.

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } 5|k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } g^k} = \begin{cases} \langle 1, x, x^2, x^3, y, \dots, y^4, xy, \dots, xy^4, x^2 y, \dots, x^2 y^4 \rangle, \mu = 16 \\ \langle 1, y, y^2, y^3 \rangle, \mu = 4 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19
$ G \cdot \deg_W$	24	18	12	6	0	28	22	16	10	38	32	26	20	48	42	36	30
invariants	$x^3 e_0$	e_1	e_2	e_3	$\mathbb{1}$	e_6	e_7	e_8	e_9	e_{11}	e_{12}	e_{13}	e_{14}	e_{16}	e_{17}	e_{18}	e_{19}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, e_1, e_{19} \rangle, \langle \mathbb{1}, e_2, e_{18} \rangle, \langle \mathbb{1}, e_3, e_{17} \rangle, \langle \mathbb{1}, \mathbb{1}, e_{16} \rangle, \langle \mathbb{1}, e_6, e_{14} \rangle, \langle \mathbb{1}, e_7, e_{13} \rangle, \langle \mathbb{1}, e_8, e_{12} \rangle$, and $\langle \mathbb{1}, e_9, e_{11} \rangle$ are all equal to 1, and $\langle x^3 e_0, x^3 e_0, \mathbb{1} \rangle = -1/4$

By the concavity axiom $\langle e_1, e_9, e_{14} \rangle, \langle e_2, e_3, e_{19} \rangle, \langle e_2, e_8, e_{14} \rangle, \langle e_2, e_9, e_{13} \rangle, \langle e_3, e_3, e_{18} \rangle, \langle e_3, e_7, e_{14} \rangle, \langle e_3, e_8, e_{13} \rangle, \langle e_3, e_9, e_{12} \rangle, \langle e_6, e_9, e_9 \rangle, \langle e_7, e_8, e_9 \rangle$, and $\langle e_8, e_8, e_8 \rangle$ are all equal to 1.

By the Index Zero axiom $\langle e_1, e_1, e_2 \rangle = -4$.

By the composition axiom $\langle x^3 e_0, e_1 = e_3 \rangle \langle x^3 e_0, e_2, e_2 \rangle = \pm 1$

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{W_{17}^T, G}$ defined by $X \mapsto e_9$ and $Y \mapsto e_3$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_9 e_3^4 = 0$, $e_3^5 = -4e_{19}$, and $e_9^3 = e_{19}$.

Hence $(4X^3 + Y^5, 5XY^4) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(4X^3 + Y^5, 5XY^4) = \mathcal{Q}$ has dimension 17, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{W_{17}, G}^T \cong \mathcal{Q}_{W_{17}}$.

$$\mathbf{Q}_{17} = \mathbf{x}^3 + \mathbf{xy}^5 + \mathbf{yz}^2.$$

$$\mathcal{J} = (3x^2 + y^5, 5xy^4 + z^2, 2yz) \quad q_x = \frac{10}{30}, q_y = \frac{4}{30}, q_z = \frac{13}{30}, \hat{c} = \frac{36}{30} \quad G = \langle J \rangle \cong \mathbb{Z}/30\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_{xy}^2 & \text{if } k = 15 \\ \mathbb{C}_x & \text{if } 3|k, k \neq 15 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, x, z, z^2, y, y^2, \dots, y^9, xy, xy^2, xy^3, xz \rangle, \mu = 17 \\ \langle 1, x, y, y^2, \dots, y^8, xy, xy^2, xy^3 \rangle, \mu = 13 \\ \langle 1, x \rangle, \mu = 2 \\ \langle 1 \rangle \end{cases}$$

k	1	2	4	5	7	8	10	11	13	14	15
$ G \cdot \deg_W$	0	54	42	36	24	18	6	60	48	42	36
invariants	$\mathbb{1}$	e_2	e_4	e_5	e_7	e_8	e_{10}	e_{11}	e_{13}	e_{14}	$y^4 e_{15}$

k	16	17	19	20	22	23	25	26	28	29
$ G \cdot \deg_W$	30	24	12	66	54	48	36	30	18	72
invariants	e_{16}	e_{17}	e_{19}	e_{20}	e_{22}	e_{23}	e_{25}	e_{26}	e_{28}	e_{29}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{29} \rangle, \langle \mathbb{1}, e_2, e_{28} \rangle, \langle \mathbb{1}, e_4, e_{26} \rangle, \langle \mathbb{1}, e_5, e_{25} \rangle, \langle \mathbb{1}, e_7, e_{23} \rangle, \langle \mathbb{1}, e_8, e_{22} \rangle, \langle \mathbb{1}, e_{10}, e_{20} \rangle, \langle \mathbb{1}, e_{11}, e_{19} \rangle, \langle \mathbb{1}, e_{13}, e_{17} \rangle$, and $\langle \mathbb{1}, e_{14}, e_{16} \rangle$ are all equal to 1, and $\langle y^4 e_{15}, y^4 e_{15}, \mathbb{1} \rangle = -1/5$

By the concavity axiom $\langle e_2, e_{10}, e_{19} \rangle, \langle e_4, e_8, e_{19} \rangle, \langle e_4, e_{10}, e_{17} \rangle, \langle e_5, e_{10}, e_{16} \rangle, \langle e_7, e_8, e_{16} \rangle, \langle e_8, e_{10}, e_{13} \rangle, \langle e_8, e_{25}, e_{28} \rangle, \langle e_{10}, e_{10}, e_{11} \rangle, \langle e_{10}, e_{23}, e_{28} \rangle, \langle e_{10}, e_{25}, e_{26} \rangle, \langle e_{16}, e_{17}, e_{28} \rangle, \langle e_{16}, e_{19}, e_{26} \rangle, \langle e_{17}, e_{19}, e_{25} \rangle, \langle e_{19}, e_{19}, e_{23} \rangle$ are all equal to 1.

By the index zero axiom $\langle e_5, e_7, e_{19} \rangle, \langle e_5, e_{28}, e_{28} \rangle, \langle e_7, e_7, e_{17} \rangle, \langle e_7, e_{10}, e_{14} \rangle, \langle e_7, e_{26}, e_{28} \rangle, \langle e_{14}, e_{19}, e_{28} \rangle$, are all equal to -2 .

By the composition axiom $\langle e_8, e_8, y^4 e_{15} \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{Q_{17}, G}$ defined by $X \mapsto e_8$ and $Y \mapsto e_{10}$ and $Z \mapsto e_{16}$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_8^2 e_{10} = 0$, $e_{10}^5 = -2e_{16}$, $e_8^3 = -5e_{22}$, and $e_{10}^4 e_{16} = e_{22}$.

Hence $(3X^2, X^3 + 5Y^4 Z, Y^5 + 2Z) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(3X^2, X^3 + 5Y^4 Z, Y^5 + 2Z) = \mathcal{Q}_{Q_{17}^T}$ has dimension 21, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Q_{17}, G} \cong \mathcal{Q}_{Q_{17}^T}$.

$$\mathbf{Q}_{17}^T = \mathbf{x}^3 \mathbf{y} + \mathbf{y}^5 \mathbf{z} + \mathbf{z}^2.$$

$$\mathcal{J} = (3x^2 y, x^3 + 5y^4 z, y^5 + 2z) \quad q_x = \frac{9}{30}, q_y = \frac{3}{30}, q_z = \frac{15}{30}, \hat{c} = \frac{36}{30} \quad G \cong \mathbb{Z}/30\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/10\mathbb{Z}$$

Since J does not generate G we will use the generator $g = (\zeta, \zeta^{-3}, \zeta^{15})$, where ζ is a primitive 30-th root of unity.

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_z & \text{if } 2|k, 10 \nmid k \\ \mathbb{C}_{yz}^2 & \text{if } 10|k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}_{|\text{Fix } g^k} = \begin{cases} \langle 1, x, x^2, z, y, \dots, y^4, xy, \dots, xy^4, xz, yz, \dots, y^4z, xyz, \dots, xy^4z \rangle \\ \langle 1 \rangle \\ \langle 1, y, z, y^2, y^3, y^4, yz, y^2z, y^3z \rangle, \mu = 9 \\ \langle 1 \rangle \end{cases}$$

k	1	3	5	7	9	10	11	13	15	17	19	20	21	23	25	27	29
$ G \cdot \deg_W$	32	24	16	8	0	26	52	44	36	28	20	46	72	64	56	48	40
invariants	e_1	e_3	e_5	e_7	$\mathbb{1}$	y^4e_{10}	e_{11}	e_{13}	e_{15}	e_{17}	e_{19}	y^4e_{20}	e_{21}	e_{23}	e_{25}	e_{27}	e_{29}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, e_1, e_{29} \rangle, \langle \mathbb{1}, e_3, e_{27} \rangle, \langle \mathbb{1}, e_5, e_{25} \rangle, \langle \mathbb{1}, e_7, e_{23} \rangle, \langle \mathbb{1}, \mathbb{1}, e_{21} \rangle, \langle \mathbb{1}, e_{11}, e_{19} \rangle, \langle \mathbb{1}, e_{13}, e_{17} \rangle$, and $\langle \mathbb{1}, e_{15}, e_{15} \rangle$ are equal to 1, and $\langle y^4e_{10}, y^4e_{20}, \mathbb{1} \rangle = -1/5$.

By the concavity axiom we see that $\langle e_1, e_{19}, e_{19} \rangle, \langle e_3, e_7, e_{29} \rangle, \langle e_3, e_{17}, e_{19} \rangle, \langle e_5, e_5, e_{29} \rangle, \langle e_5, e_7, e_{27} \rangle, \langle e_5, e_{15}, e_{19} \rangle, \langle e_5, e_{17}, e_{17} \rangle, \langle e_7, e_7, e_{25} \rangle, \langle e_7, e_{13}, e_{19} \rangle$, and $\langle e_7, e_{15}, e_{17} \rangle$ are all equal to 1.

By the Index Zero axiom $\langle e_1, e_1, e_7 \rangle = \langle e_1, e_3, e_5 \rangle = \langle e_3, e_3, e_3 \rangle = -3$.

We cannot compute the last three-point correlator $a = \langle y^4e_{10}, y^4e_{10}, e_{19} \rangle$. If $a \neq 0$, then we can establish an isomorphism between the graded rings $\mathcal{H}_{Q_{17}, G}^T$ and $\mathcal{Q}_{Q_{17}}$ in the following way:

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{Q_{17}, G}^T$ defined by $X \mapsto e_{19}$, $Y \mapsto e_7$ and $Z \mapsto \beta y^4e_{10}$ with $\beta^2 = \frac{-5}{a}$. Extending this map by \mathbb{C} -linearity and multiplicativity we get a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{19}^2 = e_{29}$, $e_7^5 = -3e_{29}$, $e_{19}e_7^4 = e_{11}$, $(\beta y^4e_{10})^2 = \beta^2(y^4e_{10})^2 = (\beta^2a)e_{11} = -5e_{11}$, and $e_7(y^4e_{10}) = 0$.

Hence $(3X^2 + Y^5, 5XY^4 + Z^2, 2YZ) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(3X^2 + Y^5, 5XY^4 + Z^2, 2YZ) = \mathcal{Q}_{Q_{17}}$ has dimension 17, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Q_{17}, G}^T \cong \mathcal{Q}_{Q_{17}}$ under the assumption that $a \neq 0$.

$$\mathbf{S}_{16} = \mathbf{x}^2\mathbf{z} + \mathbf{yz}^2 + \mathbf{xy}^4.$$

$$\mathcal{J} = (2xz + y^4, z^2 + 4xy^3, x^2 + 2yz) \quad q_x = \frac{5}{17}, q_y = \frac{3}{17}, q_z = \frac{7}{17}, \hat{c} = \frac{21}{17} \quad G = \langle J \rangle \cong \mathbb{Z}/17\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}_{|\text{Fix } J^0} = \begin{cases} \langle 1, x, y, y^2, \dots, y^6, z, z^2, z^3, xy, xy^2, yz, y^2z, y^3z \rangle, \mu = 16 \\ \langle 1 \rangle \end{cases}$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$ G \cdot \deg_W$	0	30	26	22	18	14	10	6	36	32	28	24	20	16	12	42
invariants	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{16} \rangle, \langle \mathbb{1}, e_2, e_{15} \rangle, \langle \mathbb{1}, e_3, e_{14} \rangle, \langle \mathbb{1}, e_4, e_{13} \rangle, \langle \mathbb{1}, e_5, e_{12} \rangle, \langle \mathbb{1}, e_6, e_{11} \rangle, \langle \mathbb{1}, e_7, e_{10} \rangle$, and $\langle \mathbb{1}, e_8, e_9 \rangle$ are equal to 1.

By the concavity axiom $\langle e_2, e_8, e_8 \rangle, \langle e_3, e_7, e_8 \rangle, \langle e_4, e_6, e_8 \rangle, \langle e_5, e_6, e_7 \rangle, \langle e_6, e_{14}, e_{15} \rangle, \langle e_7, e_{13}, e_{15} \rangle, \langle e_8, e_{12}, e_{15} \rangle$, and $\langle e_8, e_{13}, e_{14} \rangle$ are all equal to 1.

By the Index Zero axiom $\langle e_4, e_7, e_7 \rangle, \langle e_7, e_{14}, e_{14} \rangle, \langle e_5, e_5, e_8 \rangle, \langle e_5, e_{15}, e_{15} \rangle$ all equal to -2, and $\langle e_6, e_6, e_6 \rangle = -4$.

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{16}, G}$ defined by $X \mapsto e_7, Y \mapsto e_8$ and $Z \mapsto e_6$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_7 e_6 = e_{12}, e_8^4 = -2e_{12}, e_6^2 = -4e_{11}, e_7 e_8^3 = e_{11}, e_7^2 = -2e_{13},$ and $e_8 e_6 = e_{13}$.

Hence $(2XZ + Y^4, Z^2 + 4ZY^3, X^2 + 2YZ) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(2XZ + Y^4, Z^2 + 4ZY^3, X^2 + 2YZ) = \mathcal{Q}_{S_{16}}$ has dimension 16, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{S_{16}, G} \cong \mathcal{Q}_{S_{16}}$.

$$\mathbf{S}_{17} = \mathbf{x}^2 \mathbf{z} + \mathbf{y} \mathbf{z}^2 + \mathbf{y}^6.$$

$$\mathcal{J} = (2xy, z^2 + 6y^5, x^2 + 2yz) \quad q_x = \frac{7}{24}, q_y = \frac{4}{24}, q_z = \frac{10}{24}, \hat{c} = \frac{30}{24} \quad G = \langle J \rangle \cong \mathbb{Z}/24\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_{yz}^2 & \text{if } k = 12 \\ \mathbb{C}_y & \text{if } 6|k, k \neq 12 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, x, y, \dots, y^4, z, z^2, z^3, xy, \dots, xy^4, yz, \dots, y^4 z \rangle, \mu = 17 \\ \langle 1, y, y^2, y^3, y^4, z, z^2 \rangle, \mu = 7 \\ \langle 1, y, y^2, y^3, y^4 \rangle, \mu = 5 \\ \langle 1 \rangle \end{cases}$$

k	1	2	3	4	5	7	8	9	10	11
$ G \cdot \deg_W$	0	42	36	30	24	12	6	48	42	36
invariants	$\mathbb{1}$	e_2	e_3	e_4	e_5	e_7	e_8	e_9	e_{10}	e_{11}

k	12	13	14	15	16	17	19	20	21	22	23
$ G \cdot \deg_W$	30	24	18	12	54	48	36	30	24	42	60
invariants	ze_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{23} \rangle, \langle \mathbb{1}, e_2, e_{22} \rangle, \langle \mathbb{1}, e_3, e_{21} \rangle, \langle \mathbb{1}, e_4, e_{20} \rangle, \langle \mathbb{1}, e_5, e_{19} \rangle, \langle \mathbb{1}, e_7, e_{17} \rangle, \langle \mathbb{1}, e_8, e_{16} \rangle, \langle \mathbb{1}, e_9, e_{15} \rangle, \langle \mathbb{1}, e_{10}, e_{14} \rangle,$ and $\langle \mathbb{1}, e_{11}, e_{13} \rangle$ are equal to 1, and $\langle ze_{12}, ze_{12}, \mathbb{1} \rangle = -1/2,$

By the Concavity axiom $\langle e_2, e_8, e_{15} \rangle, \langle e_3, e_7, e_{15} \rangle, \langle e_3, e_8, e_{14} \rangle, \langle e_4, e_8, e_{13} \rangle, \langle e_5, e_7, e_{13} \rangle, \langle e_7, e_8, e_{10} \rangle, \langle e_7, e_{20}, e_{22} \rangle, \langle e_8, e_8, e_9 \rangle, \langle e_8, e_{19}, e_{22} \rangle, \langle e_8, e_{20}, e_{21} \rangle, \langle e_{13}, e_{14}, e_{22} \rangle, \langle e_{13}, e_{15}, e_{21} \rangle, \langle e_{14}, e_{15}, e_{20} \rangle, \langle e_{15}, e_{15}, e_{19} \rangle$ are all equal to 1.

By the index zero axiom, $\langle e_4, e_7, e_{14} \rangle, \langle e_5, e_5, e_{15} \rangle, \langle e_5, e_{22}, e_{22} \rangle, \langle e_7, e_7, e_{11} \rangle, \langle e_7, e_{21}, e_{21} \rangle,$ and $\langle e_{14}, e_{14}, e_{21} \rangle$ to be equal to -2

By the composition axiom $\langle e_5, e_8, ze_{12} \rangle = \langle ze_{12}, e_{15}, e_{22} \rangle = \pm 1.$

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{17}, G}$ defined by $X \mapsto e_8, Y \mapsto e_{13},$ and $Z \mapsto e_7$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{13} * e_7 = e_{19}$ and $e_8^6 = -2e_{19}, e_7^2 = -2e_{13}$ and $e_8^5 * e_7 = 0.$

Hence $(6X^5Z, Z^2 + 2Y, X^6 + 2YZ) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(6X^5Z, Z^2 + 2Y, X^6 + 2YZ) = \mathcal{Q}_{S_{17}^T}$ has dimension 21, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{S_{17}, G} \cong \mathcal{Q}_{S_{17}^T}.$

$$\mathbf{S}_{17}^T = \mathbf{x}^6 \mathbf{z} + \mathbf{z}^2 \mathbf{y} + \mathbf{y}^2.$$

$$\mathcal{J} = (6x^5z, z^2 + 2y, x^6 + 2zy) \quad q_x = \frac{3}{24}, q_y = \frac{12}{24}, q_z = \frac{6}{24}, \hat{c} = \frac{30}{24} \quad G \cong \mathbb{Z}/24\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/8\mathbb{Z}$$

Since J does not generate the symmetry group, we will use the generator $g = (\zeta, \zeta^{12}, \zeta^{-6})$ where ζ is a primitive 24-th root of unity.

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } 2 \mid k, 4 \nmid k \\ \mathbb{C}_{yz}^2 & \text{if } 4 \mid k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } g^k} = \begin{cases} \langle 1, x, x^2, \dots, x^{10}, z, z^2, xz, x^2z, x^3z, x^4z, xz^2, \dots, x^4z^2 \rangle \\ \langle 1 \rangle \\ \langle 1, z, z^2 \rangle \\ \langle 1 \rangle \end{cases}$$

k	1	3	4	5	7	8	9	11	12	13	15	16	17	19	20	21	23
$ G \cdot \deg_W$	20	0	14	28	8	22	36	16	30	44	24	38	52	32	46	60	40
invariants	e_1	$\mathbb{1}$	ze_4	e_5	e_7	ze_8	e_9	e_{11}	ze_{12}	e_{13}	e_{15}	ze_{16}	e_{17}	e_{19}	ze_{20}	e_{21}	e_{23}

Potential non-zero correlators:

By the Pairing axiom $\langle \mathbb{1}, e_1, e_{23} \rangle, \langle \mathbb{1}, \mathbb{1}, e_{21} \rangle, \langle \mathbb{1}, e_5, e_{19} \rangle, \langle \mathbb{1}, e_7, e_{17} \rangle, \langle \mathbb{1}, e_9, e_{15} \rangle$, and $\langle \mathbb{1}, e_{11}, e_{13} \rangle$ are equal to one. $\langle ze_4, ze_{20}, \mathbb{1} \rangle, \langle ze_8, ze_{16}, \mathbb{1} \rangle$, and $\langle ze_{12}, ze_{12}, \mathbb{1} \rangle$ are equal to $-1/2$.

By the Concavity axiom $\langle e_1, e_5, e_7 \rangle, \langle e_1, e_7, e_{19} \rangle, \langle e_5, e_7, e_{15} \rangle, \langle e_7, e_7, e_{13} \rangle, \langle e_7, e_9, e_{11} \rangle, \langle e_{11}, e_{11}, e_5 \rangle, \langle e_{11}, e_{15}, e_1 \rangle$, are all equal to 1.

By the index zero axiom, $\langle e_1, e_1, e_1 \rangle$ is equal to -6. The rest of the three-point correlators we cannot compute. However, if we put $c = \langle ze_4, ze_4, e_{19} \rangle, a = \langle ze_4, e_7, ze_{16} \rangle, e = \langle ze_4, ze_8, e_{15} \rangle, d = \langle ze_4, e_{11}, ze_{12} \rangle, b = \langle e_7, ze_8, ze_{12} \rangle, f = \langle ze_8, ze_8, e_{11} \rangle$ the composition axiom gives us the following relations:

$$-2af = -2bd = e \quad (4)$$

$$-2ab = d \quad (5)$$

$$-2ae = c \quad (6)$$

$$-2b^2 = f \quad (7)$$

$$-2d^2 = c \quad (8)$$

If $c \neq 0$, then we can construct an isomorphism. Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{17}, G}^T$ defined by $X \mapsto \beta ze_4$, where $\beta^2 = \frac{-2}{C}$, $Y \mapsto e_7$, and $Z \mapsto e_1$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $(ze_4)e_1 = 0, e_1^2 = -6e_{23}, e_7^5 = e_{23}, (\beta ze_4)^2 = \beta^2 C e_5 = -2e_5$ and $e_7 e_1 = e_5$.

Hence $(2XZ, Z^2 + 6Y^5, X^2 + 2YZ) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(2XZ, Z^2 + 6Y^5, X^2 + 2YZ) = \mathcal{Q}_{S_{17}}$ has dimension 17, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{S_{17}, G}^T \cong \mathcal{Q}_{S_{17}}$ if $c \neq 0$.

2.3. Singularities of Corank 3. In this section, we consider four singularities $Q_{2,0}$ and $S_{1,0}$, and the transpose of each. The other two in Arnol'd's list are not quasi-homogeneous for any choice of constants.

$$\mathbf{Q}_{2,0} = \mathbf{x}^3 + \mathbf{y}z^2 + \mathbf{x}y^4.$$

$$\mathcal{J} = (3x^2 + y^4, z^2 + 4xy^3, 2yz) \quad q_x = \frac{8}{24}, q_y = \frac{4}{24}, q_z = \frac{10}{24}, \hat{c} = \frac{28}{24} \quad G \cong \mathbb{Z}/24\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/12\mathbb{Z}$$

Since J does not generate the full group of diagonal symmetries we will use $g = (\zeta^8, \zeta^{-2}, \zeta)$, where $\zeta^{24} = 1$.

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_{xy}^2 & \text{if } k = 12 \\ \mathbb{C}_x & \text{if } 3|k, k \neq 12 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}_{\text{Fix } g^k} = \begin{cases} \langle 1, x, y, y^2 \dots, y^7, z, z^2, xy, xz \rangle, \mu = 14 \\ \langle 1, x, x^2, y, y^2, y^3, xy, xy^2, x^2y, x^2y^2 \rangle, \mu = 10 \\ \langle 1, x \rangle, \mu = 2 \\ \langle 1 \rangle \end{cases}$$

k	1	2	4	5	7	8	10	11	12	13	14	16	17	19	20	22	23
$ G \cdot \deg_W$	18	32	12	26	6	20	0	14	28	42	56	36	50	30	44	24	38
invariants	e_1	e_2	e_4	e_5	e_7	e_8	$\mathbb{1}$	e_{11}	y^3e_{12}	e_{13}	e_{14}	e_{16}	e_{17}	e_{19}	e_{20}	e_{22}	e_{23}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, e_1, e_{23} \rangle, \langle \mathbb{1}, e_2, e_{22} \rangle, \langle \mathbb{1}, e_4, e_{20} \rangle, \langle \mathbb{1}, e_5, e_{19} \rangle, \langle \mathbb{1}, e_7, e_{17} \rangle, \langle \mathbb{1}, e_8, e_{16} \rangle, \langle \mathbb{1}, e_{11}, e_{13} \rangle$, and $\langle \mathbb{1}, \mathbb{1}, e_{14} \rangle$ are all equal to 1, and $\langle \mathbb{1}, y^3e_{12}, y^3e_{12} \rangle = -1/4$.

By the Concavity Axiom $\langle e_1, e_{11}, e_{22} \rangle, \langle e_4, e_7, e_{23} \rangle, \langle e_4, e_8, e_{22} \rangle, \langle e_4, e_{11}, e_{19} \rangle, \langle e_5, e_7, e_{22} \rangle, \langle e_7, e_7, e_{20} \rangle, \langle e_7, e_8, e_{19} \rangle, \langle e_7, e_{11}, e_{16} \rangle$ and are all equal to 1.

By the Index Zero Axiom $\langle e_1, e_2, e_7 \rangle = -2$

By the Composition Axiom $\langle e_{11}, e_{11}, y^3e_{12} \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{Q_{2,0}, G}$ defined by $X \mapsto e_{11}$, $Y \mapsto e_7$ and $Z \mapsto e_{22}$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{11}^2e_7 = 0$, $e_{11}^3 = -4e_{13}$, $e_7^3e_{22} = e_{13}$, and that $e_7^4 = -2e_{22}$.

Hence $(3X^2Y, X^3 + 4Y^3Z, 2Z + Y^4) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(3X^2Y, X^3 + 4Y^3Z, 2Z + Y^4) = \mathcal{Q}_{Q_{2,0}}^T$ has dimension 17, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Q_{2,0}, G} \cong \mathcal{Q}_{Q_{2,0}}^T$.

$$\mathbf{Q}_{2,0}^T = \mathbf{x}^3\mathbf{y} + \mathbf{y}^4\mathbf{z} + \mathbf{z}^2.$$

$$\mathcal{J} = (3x^2y, x^3 + 4y^3z, 2z + y^4) \quad q_x = \frac{7}{24}, q_y = \frac{3}{24}, q_z = \frac{12}{24}, \hat{c} = \frac{28}{24} \quad G = \langle J \rangle \cong \mathbb{Z}/24\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_{yz}^2 & \text{if } 8|k \\ \mathbb{C}_z & \text{if } 2|k, 8 \nmid k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}_{\text{Fix } J^k} = \begin{cases} \langle 1, x, x^2, y, y^2, \dots, y^7, xy, xy^2, \dots, xy^7 \rangle, \mu = 17 \\ \langle 1 \rangle \\ \langle 1, y, y^2, y^3, z, yz, y^2z \rangle, \mu = 7 \\ \langle 1 \rangle \end{cases}$$

k	1	3	5	7	8	9	11	13	15	16	17	19	21	23
$ G \cdot \deg_W$	0	40	32	24	20	16	8	48	40	36	32	24	16	56
invariants	$\mathbb{1}$	e_3	e_5	e_7	y^3e_8	e_9	e_{11}	e_{13}	e_{15}	y^3e_{16}	e_{17}	e_{19}	e_{21}	e_{23}

Potential non-zero correlators:

By the paring axiom $\langle \mathbb{1}, \mathbb{1}, e_{23} \rangle, \langle \mathbb{1}, e_3, e_{21} \rangle, \langle \mathbb{1}, e_5, e_{19} \rangle, \langle \mathbb{1}, e_7, e_{17} \rangle, \langle \mathbb{1}, e_9, e_{15} \rangle$, and $\langle \mathbb{1}, e_{11}, e_{13} \rangle$ are equal to 1, and $\langle y^3e_8, y^3e_{16}, \mathbb{1} \rangle = -1/4$.

By the concavity axiom $\langle e_3, e_{11}, e_{11} \rangle, \langle e_5, e_9, e_{11} \rangle, \langle e_7, e_9, e_9 \rangle, \langle e_9, e_{19}, e_{21} \rangle, \langle e_{11}, e_{17}, e_{21} \rangle$, and $\langle e_{11}, e_{19}, e_{19} \rangle$ are equal to 1.

By the Index Zero axiom $\langle e_7, e_7, e_{11} \rangle = \langle e_7, e_{21}, e_{21} \rangle = -3$

The axioms do not provide us an easy way to calculate the last three-point correlator, $a = \langle y^3 e_8, y^3 e_8, e_9 \rangle$. If $a \neq 0$, then we can construct the desired homomorphism as follows:

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{Q_{2,0}, G}$ defined by $X \mapsto e_9$, where $\beta^2 = \frac{-4}{a}$, $Y \mapsto e_{11}$ and $Z \mapsto \beta y^3 e_8$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_{11}(y^3 e_8) = 0$, $e_9^2 = e_{17}$, $e_{11}^4 = -3e_{17}$, $(\beta y^3 e_8)^2 = (\beta^2 a)e_{15} = -4e_{15}$ and that $e_9 e_{11}^3 = e_{15}$.

Hence $(3X^2 + Y^4, 2YZ, Z^2 + 4XY^3) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(3X^2 + Y^4, 2YZ, Z^2 + 4XY^3) = \mathcal{Q}_{Q_{2,0}}$ has dimension 14, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{Q_{2,0}, G} \cong \mathcal{Q}_{Q_{2,0}}$, assuming $a \neq 0$.

$$\mathbf{S}_{1,0} = \mathbf{x}^2 \mathbf{z} + \mathbf{y} \mathbf{z}^2 + \mathbf{y}^5.$$

$$\mathcal{J} = (2xz, z^2 + 5y^4, x^2 + 2yz) \quad q_x = \frac{6}{20}, q_y = \frac{4}{20}, q_z = \frac{8}{20}, \hat{c} = \frac{24}{20} \quad G \cong \mathbb{Z}/20\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/10\mathbb{Z}$$

Since J does not generate the full group of diagonal symmetries we will use $g = (\zeta, \zeta^4, \zeta^{-2})$, where $\zeta^{20} = 1$.

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_{yz}^2 & \text{if } k = 10 \\ \mathbb{C}_y & \text{if } 5|k, k \neq 10 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } g^k} = \begin{cases} \langle 1, x, y, y^2, y^3, z, z^2, z^3, xy, xy^2, y^3, yz, y^2 z, y^3 z \rangle, \mu = 14 \\ \langle 1, y, y^2, y^3, y^4, z \rangle, \mu = 6 \\ \langle 1, y, y^2, y^3 \rangle, \mu = 4 \\ \langle 1 \rangle \end{cases}$$

k	1	2	3	4	6	7	8	9	10	11	12	13	14	16	17	18	19
$ G \cdot \deg_W$	10	16	22	28	0	6	12	18	24	30	36	42	48	20	26	32	38
invariants	e_1	e_2	e_3	e_4	$\mathbb{1}$	e_7	e_8	e_9	ze_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{16}	e_{17}	e_{18}	e_{19}

Potential non-zero correlators:

By the Pairing Axiom $\langle \mathbb{1}, e_1, e_{19} \rangle, \langle \mathbb{1}, e_2, e_{18} \rangle, \langle \mathbb{1}, e_3, e_{17} \rangle, \langle \mathbb{1}, e_4, e_{16} \rangle, \langle \mathbb{1}, \mathbb{1}, e_{14} \rangle, \langle \mathbb{1}, e_7, e_{13} \rangle, \langle \mathbb{1}, e_8, e_{12} \rangle$, and $\langle \mathbb{1}, e_9, e_{11} \rangle$ are equal to 1, and $\langle \mathbb{1}, ze_{10}, ze_{10} \rangle = -1/2$.

By the Concavity Axiom $\langle e_1, e_7, e_{18} \rangle, \langle e_1, e_8, e_{17} \rangle, \langle e_1, e_9, e_{16} \rangle, \langle e_2, e_7, e_{17} \rangle, \langle e_2, e_8, e_{16} \rangle, \langle e_3, e_7, e_{16} \rangle, \langle e_7, e_7, e_{12} \rangle$, and $\langle e_7, e_8, e_{11} \rangle$ are all equal to 1.

By the Index Zero Axiom $\langle e_1, e_1, e_4 \rangle, \langle e_1, e_2, e_3 \rangle, \langle e_2, e_2, e_2 \rangle, \langle e_8, e_9, e_9 \rangle$ are equal to -2.

By the Composition Axiom $\langle e_7, e_9, ze_{10} \rangle = \langle e_8, e_8, ze_{10} \rangle = \pm 1$.

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{1,0}, G}$ defined by $X \mapsto e_{16}$, $Y \mapsto e_1$ and $Z \mapsto e_7$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $e_1 e_7^4 = 0$, $e_1^2 = -2e_{16}$, $e_{16} e_1 = e_{11}$, and that $e_7^5 = -2e_{11}$.

Hence $(2X + Y^2, Z^5 + 2XY, 5YZ^4) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(2X + Y^2, Z^5 + 2XY, 5YZ^4) = \mathcal{Q}_{S_{1,0}^T}$ has dimension 17, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{S_{1,0}, G} \cong \mathcal{Q}_{S_{1,0}^T}$.

$$\mathbf{S}_{1,0}^T = \mathbf{x}^2 + \mathbf{y} \mathbf{z}^5 + \mathbf{x} \mathbf{y}^2.$$

$$\mathcal{J} = (2x + y^2, z^5 + 2xy, 5yz^4) \quad q_x = \frac{10}{20}, q_y = \frac{5}{20}, q_z = \frac{3}{20}, \hat{c} = \frac{24}{20} \quad G = \langle J \rangle \cong \mathbb{Z}/20\mathbb{Z}$$

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^3 & \text{if } k = 0 \\ \mathbb{C}_{xy}^2 & \text{if } 4|k \\ \mathbb{C}_x & \text{if } 2|k, 4 \nmid k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, y, y^2, z, z^2, \dots, z^8, yz, yz^2, yz^3, y^2z, y^2z^2, y^2z^3 \rangle, \mu = 17 \\ \langle 1 \rangle \\ \langle 1, y, y^2 \rangle, \mu = 3 \\ \langle 1 \rangle \end{cases}$$

k	1	3	4	5	7	8	9	11	12	13	15	16	17	19
$ G \cdot \deg_W$	0	32	28	24	16	12	8	40	36	32	24	20	16	48
invariants	$\mathbb{1}$	e_3	ye_4	e_5	e_7	ye_8	e_9	e_{11}	ye_{12}	e_{13}	e_{15}	ye_{16}	e_{17}	e_{19}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{19} \rangle$, $\langle \mathbb{1}, e_3, e_{17} \rangle$, $\langle \mathbb{1}, e_5, e_{15} \rangle$, $\langle \mathbb{1}, e_7, e_{13} \rangle$, and $\langle \mathbb{1}, e_9, e_{11} \rangle$ are equal to 1, and $\langle ye_4, ye_{16}, \mathbb{1} \rangle = \langle ye_8, ye_{12}, \mathbb{1} \rangle = -1/2$.

By the concavity axiom $\langle e_3, e_9, e_9 \rangle$, $\langle e_5, e_7, e_9 \rangle$, $\langle e_7, e_{17}, e_{17} \rangle$, and $\langle e_9, e_{15}, e_{17} \rangle$ are equal to 1.

By the Index Zero axiom $\langle e_7, e_7, e_7 \rangle = -5$,

The composition axiom does not allow us to compute the last four three-point correlators, however if we let $a = \langle ye_4, e_8, e_9 \rangle$, $b = \langle e_5, ye_8, ye_8 \rangle$, $c = \langle ye_8, ye_{16}, e_{17} \rangle$, and $d = \langle ye_{16}, ye_{16}, e_9 \rangle$, we get the following relations:

$$\begin{aligned} b &= -2ac \\ c &= -2ad \end{aligned}$$

If $b \neq 0$, then we can construct the desired isomorphism.

Consider the map $\varphi : \mathbb{C}[X, Y, Z] \rightarrow \mathcal{H}_{S_{1,0}, G}^T$ defined by $X \mapsto \beta y^3 e_8$, where $\beta^2 = \frac{-2}{b}$, $Y \mapsto e_9$ and $Z \mapsto e_7$ and extending to a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Our correlators tell us that $(y^3 e_8) e_7 = 0$, $e_7^2 = -5e_{13}$, $e_9^4 = e_{13}$, $(\beta y^3 e_8)^2 = (\beta^2 b) e_{15} = -2e_{15}$ and that $e_9 e_7 = e_{15}$.

Hence $(2XZ, Z^2 + 5Y^4, X^2 + 2YZ) \subset \ker \varphi$. Since $\mathbb{C}[X, Y, Z]/(2XZ, Z^2 + 5Y^4, X^2 + 2YZ) = \mathcal{Q}_{S_{1,0}}$ has dimension 14, we deduce that the inclusion is equality, and so we have the isomorphism $\mathcal{H}_{S_{1,0}, G}^T \cong \mathcal{Q}_{S_{1,0}}$ assuming $b \neq 0$.

2.4. Singularities of Corank 2. Here we consider only three singularities of the four singularities listed by Arnol'd, namely $J_{3,0}$, $Z_{1,0}$ and $W_{1,0}$. The singularity $W_{1,2q}^\#$ is not quasi-homogeneous for any choice of constants.

$$\mathbf{J}_{3,0} = \mathbf{x}^3 + \mathbf{b}\mathbf{x}^2\mathbf{y}^3 + \mathbf{y}^9, 4\mathbf{b}^3 + 27 \neq 0.$$

$$\mathcal{J} = (3x^2 + 2bxy^3, 3bx^2y^2 + 9y^8) \quad q_x = \frac{3}{9}, \quad q_y = \frac{1}{9}, \quad \hat{c} = \frac{10}{9} \quad G = \langle J \rangle \cong \mathbb{Z}/9\mathbb{Z}$$

Notice that if $b = 0$, then the maximal group of diagonal symmetries is not cyclic. But in that case $\mathcal{H}_{Z_{3,0}, G}|_{b=0}$ is isomorphic to a tensor product of simple singularities. So in this paper we only consider the case when the admissible group is generated by J .

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_x & \text{if } 3|k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, x, y, y^2, \dots, y^7, xy, xy^2, \dots, xy^7 \rangle, \mu = 16 \\ \langle 1, x \rangle, \mu = 2 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	4	5	7	8
$ G \cdot \deg_W$	10	0	8	6	14	12	20
invariants	$y^5 e_0, xy^2 e_0$	$\mathbb{1}$	e_2	e_4	e_5	e_7	e_8

Potential non-zero correlators:

By of the pairing axiom $\langle \mathbb{1}, e_8 \rangle = \langle \mathbb{1}, e_2, e_7 \rangle = \langle \mathbb{1}, e_4, e_5 \rangle = 1$, and

$$\begin{aligned}\langle y^5 e_0, y^5 e_0, \mathbb{1} \rangle &= \frac{2b^2}{9(4b^3 + 27)} \\ \langle y^5 e_0, xy^2 e_0, \mathbb{1} \rangle &= \frac{1}{4b^3 + 27} \\ \langle xy^2 e_0, xy^2 e_0, \mathbb{1} \rangle &= \frac{-2b}{3(4b^3 + 27)}\end{aligned}$$

By the concavity axiom $\langle e_2, e_4, e_4 \rangle = 1$

Notice that in this case, we have the interesting situation that $b = 0$ gives a different *FJRW* ring for this singularity, than any other value for b .

Another interesting observation is that in this case, there is no Milnor ring that is isomorphic to the FJRW ring. To see this suppose there is a quasi-homogeneous polynomial $f(x_1, x_2, x_3, x_4)$ with

$$\mathbb{C}[x_1, x_2, x_3, x_4]/\mathcal{J}_f \cong \mathcal{H}_{J_{3,0}, \langle J \rangle}$$

and let q_1, q_2, q_3, q_4 be the corresponding charges. The isomorphism must send generators to generators, so we may assume $x_1 \mapsto \mu$, $x_2 \mapsto \nu$, $x_3 \mapsto e_2$, and $x_4 \mapsto e_4$, where μ and ν are in the untwisted sector. Since we have an isomorphism of graded rings,

$$q_1 = 10\alpha, \quad q_2 = 10\alpha, \quad q_3 = 8\alpha, \quad q_4 = 6\alpha, \quad \hat{c}_f = 20\alpha$$

(We admit the possibility that the grading may differ by a uniform scaling, although we see below that this possibility does not occur).

From the definition of \hat{c}_f , we have

$$\hat{c}_f = \sum (1 - 2q_i) = 4 - 68\alpha,$$

which we can solve to find $\alpha = \frac{1}{22}$, so the weights are

$$q_1 = \frac{5}{11}, \quad q_2 = \frac{5}{11}, \quad q_3 = \frac{4}{11}, \quad q_4 = \frac{3}{11}$$

Now this Milnor ring must have dimension $\mu = 8$, but the dimension is given in terms of the charges by

$$\mu = \prod \left(\frac{1}{q_i} - 1 \right) = \frac{168}{25},$$

a contradiction.

$$\mathbf{Z}_{1,0} = \mathbf{x}^3 \mathbf{y} + \mathbf{y}^7.$$

$$\mathcal{J} = (3x^2 y, x^3 + 7y^6) \quad q_x = \frac{2}{7}, \quad q_y = \frac{1}{7}, \quad \hat{c} = \frac{8}{7} \quad G \cong \mathbb{Z}/21\mathbb{Z} \quad \langle J \rangle \cong \mathbb{Z}/7\mathbb{Z}$$

$$\text{Fix } g^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C}_y & \text{if } 7|k, k \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } g^k} = \begin{cases} \langle 1, x, x^2, y, \dots, y^6, xy, xy^2, \dots, xy^6 \rangle, \mu = 21 \\ \langle 1, y, y^2, y^3, y^4, y^5 \rangle, \mu = 6 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	3	4	5	6	8	9	10	11	12	13	15	16	17	18	19	20
$ G \cdot \deg_W$	24	20	16	12	8	4	0	34	30	26	22	18	14	48	44	40	36	32	28
invariants	$x^2 e_0$	e_1	e_2	e_3	e_4	e_5	$\mathbb{1}$	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{15}	e_{16}	e_{17}	e_{18}	e_{19}	e_{20}

Potential non-zero correlators:

By the pairing axiom $\langle \mathbb{1}, e_1, e_{20} \rangle, \langle \mathbb{1}, e_2, e_{19} \rangle, \langle \mathbb{1}, e_3, e_{18} \rangle, \langle \mathbb{1}, e_4, e_{17} \rangle, \langle \mathbb{1}, e_5, e_{16} \rangle, \langle \mathbb{1}, \mathbb{1}, e_{15} \rangle, \langle \mathbb{1}, e_8, e_{13} \rangle, \langle \mathbb{1}, e_9, e_{12} \rangle$, and $\langle \mathbb{1}, e_{10}, e_{11} \rangle$ are all equal to 1 and $\langle x^2 e_0, x^2 e_0, \mathbb{1} \rangle = -1/3$.

By the concavity axiom we have the following 3-point correlators are equal to 1: $\langle e_1, e_{13}, e_{13} \rangle, \langle e_2, e_5, e_{20} \rangle, \langle e_2, e_{12}, e_{13} \rangle, \langle e_3, e_5, e_{19} \rangle, \langle e_3, e_4, e_{20} \rangle, \langle e_3, e_{11}, e_{13} \rangle, \langle e_3, e_{12}, e_{12} \rangle, \langle e_4, e_4, e_{19} \rangle, \langle e_4, e_5, e_{18} \rangle, \langle e_4, e_{10}, e_{13} \rangle, \langle e_4, e_{11}, e_{12} \rangle, \langle e_5, e_5, e_{17} \rangle, \langle e_5, e_9, e_{13} \rangle, \langle e_5, e_{10}, e_{12} \rangle$, and $\langle e_5, e_{11}, e_{11} \rangle$.

By the index zero axiom $\langle e_1, e_1, e_4 \rangle, \langle e_1, e_2, e_3 \rangle$, and $\langle e_2, e_2, e_2 \rangle$ are all equal to -3.

By the composition axiom using an argument similar that that for equation (3) we have $\langle x^2 e_0, e_1, e_5 \rangle = \langle x^2 e_0, e_2, e_4 \rangle = \langle x^2 e_0, e_3, e_3 \rangle = \pm 1$

Again, by examining degrees, we see that e_5 and e_{13} are generators for $\mathcal{H}_{J_{1,0},G}$.

Consider the map $\varphi : \mathbb{C}[X, Y] \rightarrow \mathcal{H}_{J_{1,0},G}$ defined by $X \mapsto e_{13}$ and $Y \mapsto e_5$ and extending as a \mathbb{C} -algebra homomorphism. One can check directly that this map is surjective.

Straightforward computations show that $e_5^7 = -3e_{20}$, $e_{13}^2 = e_{20}$, and $e_{13}e_5^6 = 0$.

Hence $(Y^7 + 3X^2, XY^6) \subset \ker \varphi$. Since $\mathbb{C}[X, Y]/(Y^7 + 3X^2, XY^6) = \mathcal{Q}_{Z_{1,0}}$ has dimension 19, we deduce that the inclusion is equality. The reader will note that $Z_{1,0}$ is the transposed singularity for E_{19} which we computed as an example in the introduction, so we have the isomorphism $\mathcal{Q}_{E_{19}} \cong \mathcal{H}_{J_{1,0},G}$.

$\mathbf{W}_{1,0} = \mathbf{x}^4 + \mathbf{a}\mathbf{x}^2\mathbf{y}^3 + \mathbf{y}^6, \mathbf{a}^2 \neq 4$.

$$\mathcal{J} = (4x^3 + 2axy^3, 3ax^2y^2 + 6y^5) \quad q_x = \frac{3}{12}, \quad q_y = \frac{2}{12}, \quad \hat{c} = \frac{14}{12} \quad G = \langle J \rangle \cong \mathbb{Z}/12\mathbb{Z}$$

Just as with $J_{3,0}$, if $a = 0$, then the maximal group of diagonal symmetries is not cyclic. But in that case $\mathcal{H}_{Z_{3,0},G}|_{b=0}$ is isomorphic to a tensor product of simple singularities. So in this paper we only consider the case when the admissible group is generated by J .

$$\text{Fix } J^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ C_x & \text{if } 4|k \\ C_y & \text{if } 3|k \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{Q}|_{\text{Fix } J^k} = \begin{cases} \langle 1, x, x^2, y, \dots, y^4, xy, \dots, xy^4, x^2y, \dots, x^2y^4 \rangle, \mu = 15 \\ \langle 1, x, x^2 \rangle, \mu = 3 \\ \langle 1, y, y^2, y^3, y^4 \rangle, \mu = 5 \\ \langle 1 \rangle \end{cases}$$

k	0	1	2	5	7	10	11
$ G \cdot \deg_W$	14	0	10	16	12	18	28
invariants	xy^2e_0	$\mathbb{1}$	e_2	e_5	e_7	e_{10}	e_{11}

Potential non-zero correlators:

By the Pairing axiom $\langle \mathbb{1}, \mathbb{1}, e_{11} \rangle, \langle \mathbb{1}, e_2, e_{10} \rangle, \langle \mathbb{1}, e_3, e_9 \rangle, \langle \mathbb{1}, e_4, e_8 \rangle, \langle \mathbb{1}, e_5, e_7 \rangle$, and $\langle \mathbb{1}, e_6, e_6 \rangle$ are equal to 1 and $\langle xy^2 e_0, xy^2 e_0, \mathbb{1} \rangle = \frac{1}{24-6a^2}$

By the Concavity axiom $\langle e_2, e_2, e_9 \rangle, \langle e_7, e_9, e_9 \rangle$, and $\langle e_8, e_8, e_9 \rangle$ are equal to 1.

Just as with $J_{3,0}$, there is no Milnor ring isomorphic to this FJRW-ring. To see this, suppose there exists a quasi-homogeneous polynomial $f(x_1, x_2, \dots, x_5)$ with

$$\mathbb{C}[x_1, x_2, \dots, x_5]/\mathcal{J}_f \cong \mathcal{H}_{W_{1,0}, \langle J \rangle}$$

and let q_1, q_2, \dots, q_5 be the charges for x_1, x_2, \dots, x_5 resp. The isomorphism must send generators to generators, so we may assume $x_1 \mapsto (xy^2 e_0)$, $x_2 \mapsto e_2$, $x_3 \mapsto e_5$, $x_4 \mapsto e_7$, and $x_5 \mapsto e_{10}$. Since we have an

isomorphism of graded rings (again allowing for a uniform rescaling of degrees),

$$q_1 = 14\alpha, \quad q_2 = 10\alpha, \quad q_3 = 16\alpha, \quad q_4 = 12\alpha, \quad q_5 = 18\alpha, \quad \hat{c}_f = 28\alpha$$

From the definition of \hat{c}_f , we have

$$\hat{c}_f = \sum (1 - 2q_i) = 5 - 140\alpha,$$

Which we can solve to find $\alpha = \frac{5}{168}$, so the weights are

$$q_1 = \frac{1}{12}, \quad q_2 = \frac{5}{84}, \quad q_3 = \frac{10}{21}, \quad q_4 = \frac{5}{14}, \quad q_5 = \frac{15}{28}.$$

The dimension of the Milnor ring is given in terms of the charges by

$$\mu = \prod \left(\frac{1}{q_i} - 1 \right),$$

which is not even an integer. So the FJRW A-model in this case is not isomorphic to the Milnor ring of a quasi-homogeneous polynomial.

REFERENCES

- [1] ARNOL'D, V., GUSEIN-ZADE, S., AND VARCHENKO, A. *Singularities of Differentiable Maps Vols I, II*. Birkhauser, 1985.
- [2] BERGLUND, P., AND HUBSCH, T. A generalized construction of mirror manifolds. *Nuclear Physics B* 393 (1993), 377.
- [3] FAN, H., JARVIS, T. J., AND RUAN, Y. Personal communication.
- [4] FAN, H., JARVIS, T. J., AND RUAN, Y. The witten equation, mirror symmetry and quantum singularity theory. arXiv:0712.4021v3 [math.AG], January 2009.

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